Multiple resonance networks with incomplete energy transfer and operating with zero-state response

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***Abstract*—This paper discusses two closely related classes of linear "multiple resonance networks". The first class, instead of transferring all the energy in an input capacitor or inductor to the output capacitance of the network, leaves part of the energy in the input circuit. The other class operates with zero-state response, accumulating energy coming from an input sinusoidal voltage source, and at a certain instant delivering it completely to the output capacitance of the network. It is shown that the second class can be directly derived from the first class. Both kinds of networks find applications in high-voltage generators for pulsed power applications.**

I. INTRODUCTION

The basic concepts about "multiple resonance networks" were described in [1]-[6]. In all the cases so far studied, these networks comprise an LC ladder network, where some initial energy stored in an input capacitor [1]-[3] or inductor [4] at the input end of the network, is completely transferred, through a transient involving oscillations at multiple frequencies, to an output capacitance or inductance in the other end of the ladder. These circuits are then generalizations of the classical "Tesla transformer" and "induction coil", with applications especially in the generation of high-voltage pulses. Versions with high-pass sections were also described [5][6]. In all cases, it was shown that a very simple synthesis method can be applied, that reduces the problem to the synthesis of a certain LC impedance, as seen from the output side of the network [2][4][5][6]. Tesla transformers with incomplete energy transfer find some application when it's desired to obtain a somewhat larger voltage gain without redesigning the inductive elements [7][8][9], at the expense of some efficiency. This work starts by deriving a network synthesis solution to this last problem, that is easily extended to different networks, and that as a byproduct generates a different class of multiple resonance networks, that instead of starting with a fixed amount of energy in an element, are initially at zero state and receive energy continuously from an input sinusoidal source, concentrating it after some time at the output capacitance.

II. MULTIPLE RESONANCE NETWORKS WITH INCOMPLETE ENERGY TRANSFER BETWEEN CAPACITANCES

Consider the case of a low-pass multiple resonance network (fig. 1), where the objective is to transfer the energy initially stored in C_1 to C_p, at the closing of the switch. Due to the structure of the network, the output voltage generated from an input voltage in C_1 must have the form (1) in Laplace transform [2], where the k_j are constants that multiply a "base" frequency ω_0. This expression can be expanded in partial fractions, as a sum of cosinusoid transforms (2). For complete energy transfer to be possible, all the k_j must be positive integers, with odd differences. This guarantees that after some cycles, in $t = \pi/\omega_0$, all the cosinusoids that form the output are in phase, and the output voltage is maximum.

$$V_{out}(s) = \frac{\alpha s}{\left(s^2 + k_1^2\omega_0^2\right)\left(s^2 + k_2^2\omega_0^2\right)\cdots\left(s^2 + k_p^2\omega_0^2\right)} \quad (1)$$

$$V_{out}(s) = \sum_{j=1}^{p} \frac{A_j s}{s^2 + k_j^2\omega_0^2} \quad (2)$$

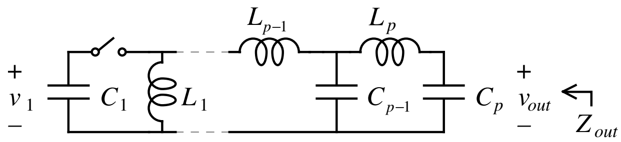

Figure 1. Low-pass transformerless multiple resonance network.

Possible values for the residues A_j in (2) can be calculated by expanding (1) with $\alpha = 1$. The maximum output voltage is then simply the sum of the absolute values of the A_j. If complete energy transfer occurs, and if the origin of time is considered at $t = \pi/\omega_0$, we have the same output waveform (2), but with all the A_j in absolute value, generated by an input voltage in C_p only. The same result is obtained with a unit impulse current source applied to the output, what means that the new output voltage is proportional to the output impedance of the network, $Z_{out}(s)$. The network that results in complete energy transfer can be obtained, with an impedance scaling factor, by simply expanding the impedance (3) in Cauer's first form [2]. A conveniently normalized network, with C_p = 1 F, is obtained by scaling the A_j so the sum of their absolute values is 1.

$$Z_{out}(s) = \sum_{j=1}^{p} \frac{|A_j| s}{s^2 + k_j^2\omega_0^2} \quad (3)$$

If we consider that at $t = \pi/\omega_0$ there is some energy left in C_1, and only there and in C_p, and the origin of time is shifted to this instant, the output waveform becomes caused by the voltage in C_p, that introduces a component proportional to $Z_{out}(s)$, and by the remaining voltage in C_1, that introduces a component with the same shape of (2). Let's assume that the component caused by the remaining charge in C_1 is multiplied by ε and that the component caused by the charge in C_p, is identical to a new, unknown, $Z_{out}(s)$. If the output voltage is the same of the case with complete energy transfer:

$$V_{out}(s) = \sum_{j=1}^{p} \frac{|A_j| s}{s^2 + k_j^2\omega_0^2} = Z_{out}(s) + \varepsilon \sum_{j=1}^{p} \frac{A_j s}{s^2 + k_j^2\omega_0^2} \quad (4)$$

And so $Z_{out}(s)$ can be directly obtained as (5), and the network obtained exactly as in the previous case.

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$$Z_{out}(s) = \sum_{j=1}^{p} \frac{\left(\left|A_j\right| - \varepsilon A_j\right)s}{s^2 + k_j^{\,2}\omega_0^{\,2}} \tag{5}$$

Note that if the A_j are scaled so the $|A_j|$ add to 1, the terms $|A_j|-\varepsilon A_j$ also add to 1, because the A_j add to 0. The expansion of $Z_{out}(s)$ with the structure in fig. 1 always starts with $C_p = 1$ F. Any value of ε between ±1 results in a realizable network. There is no apparent advantage in using this design for a transformerless network, however. It is observed that the voltage gain is simply increased by the factor ε, in relation to the voltage gain obtained with $\varepsilon = 0$.

Example 1

The design for a Tesla transformer in [7][8] can be obtained by designing the transformerless circuit in fig. 2a in a way that maximizes the ratio between the obtained voltage gain and the voltage gain that would result with the circuit tuned for complete energy transfer, but using the same inductors of the modified design. For the fastest mode 1:2, the voltage gain with complete energy transfer is 5/3, and the gain increase factor is:

$$A = \left(\frac{5}{3} + \varepsilon\right)\sqrt{\frac{L_2}{L_2 + L_1}} \tag{6}$$

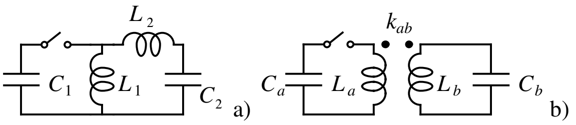

Figure 2. a) 4th order multiple resonance network. b) Tesla transformer.

With $C_2 = 1$ F and resonances at 1 and 2 rad/s, from (1), (2), and (5), the element values for fig. 2a result as:

$$C_2 = 1;\;\; L_2 = \frac{2}{2\varepsilon + 5};\;\; C_1 = \frac{(3\varepsilon + 5)^2}{9(1-\varepsilon^2)};\;\; L_1 = \frac{9(1-\varepsilon^2)}{8(3\varepsilon + 5)} \tag{7}$$

The gain increase factor (6) is then:

$$A = (3\varepsilon + 5)\sqrt{\frac{\varepsilon^2 - 1}{9\varepsilon^2 - 25}} \tag{8}$$

The value of ε that maximizes this factor is found as:

$$\varepsilon = \frac{5}{9} - \frac{20}{9} sin\left(\frac{1}{3}\tan^{-1}\frac{2\sqrt{69}}{207}\right) = 0.4962399493 \tag{9}$$

This ε results in the same gain and same tuning relations deduced in [7][8][9], by a different method. Using the relations between the circuits in fig. 2a and in fig. 2b, from (7)-(9):

$$\begin{aligned} T &= \frac{L_b C_b}{L_a C_a} = \frac{(L_1 + L_2)C_2}{L_1 C_1} = 0.5411360382 \\ k_{ab} &= \sqrt{\frac{L_1}{L_1 + L_2}} = 0.5456592065;\quad A = 1.1802099077 \end{aligned} \tag{10}$$

III. Operation with Zero-State Response - Cosinusoidal Input

When $\varepsilon = \pm 1$ in (5), some of the terms of the partial fraction expansion disappear (half of them for even number of resonances, and half ± 1 for odd number). Before disappearing, the terms reduce to low impedance LC tanks, which act as cosinusoidal oscillators driving the remaining network. In the 4th-order case, the resulting circuit when $\varepsilon = \pm 1$ is equivalent to the replacement of the tank L_1C_1 in fig. 2a by a cosinusoidal voltage source. This case is of little interest, however. In the 6th-order case, $\varepsilon = 1$ causes the disappearance of two terms, and leaves only the last LC section in fig. 1. This case is also uninteresting. $\varepsilon = -1$ produces an interesting network (fig. 3), where the tank L_1C_1 in fig. 1 reduces to a cosinusoidal voltage source, followed by a 4th-order low-pass section. This circuit can be designed by expanding the output impedance (5) for a chosen mode $k_1{:}k_2{:}k_3$.

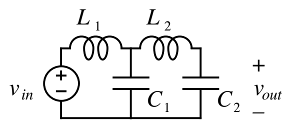

Figure 3. Double resonance low-pass multiple resonance network with zero-state response (elements renumbered).

This structure, however, is not very interesting, because there is no way to insert a transformer on it. A more interesting network is obtained when the same idea is applied to a band-pass multiple resonance network [6]. The only differences are that the numerator of (1) has a term αs^3 instead of αs, and that the expansion of $Z_{out}(s)$ takes the form in fig. 4a.

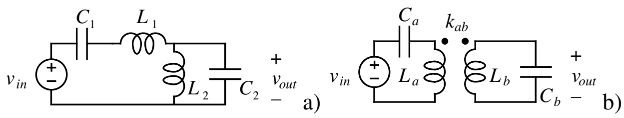

Figure 4. a) Double resonance band-pass multiple resonance network with zero-state response. b) Version with transformer.

For mode $k{:}l{:}m$, three integers with odd differences (or multiples of them), with $\omega_0 = 1$ rad/s and $C_2 = 1$ F, the obtained residues, output impedance, and network elements are:

$$\begin{aligned} &V_{out}(s) = \frac{\alpha s^3}{(s^2+k^2)(s^2+l^2)(s^2+m^2)} = \frac{A_1 s}{s^2+k^2} + \frac{A_2 s}{s^2+l^2} + \frac{A_3 s}{s^2+m^2}; \\ &A_1 = -\frac{k^2(l^2-m^2)}{2l^2(k^2-m^2)};\;\; A_2 = \frac{1}{2};\;\; A_3 = -\frac{m^2(k^2-l^2)}{2l^2(k^2-m^2)}; \\ &Z_{out}(s) = -\frac{2A_1 s}{s^2+k^2} - \frac{2A_3 s}{s^2+m^2} = \frac{s^3 + (k^2m^2/l^2)s}{(s^2+k^2)(s^2+m^2)}; \\ &C_2 = 1;\;\; L_2 = \frac{1}{l^2};\;\; L_1 = \frac{l^2}{(k^2-l^2)(l^2-m^2)};\;\; C_1 = \frac{(k^2-l^2)(l^2-m^2)}{k^2m^2} \end{aligned} \tag{11}$$

This circuit can be easily converted to a version with transformer (fig. 4b) by using the equations, with an adequate ω_p:

$$\begin{aligned} &\omega_p C_a L_a = C_1(L_1 + L_2);\;\; \omega_p C_b L_b = C_2 L_2; \\ &k_{ab} = \sqrt{\frac{L_2}{L_1 + L_2}} \end{aligned} \tag{12}$$

Further analysis reveals that this circuit has some interesting advantages over the conventional Tesla transformer. The first is that the voltage gain can be easily much higher. It can be shown to be:

$$A_v = \sqrt{\frac{C_a}{C_b}}\frac{2km}{\sqrt{(k^2-l^2)(l^2-m^2)}} \tag{13}$$

The voltage gain of a Tesla transformer is just $(C_a/C_b)^{1/2}$. Another advantage is that the primary capacitor doesn't have to hold all the final output energy. For modes $k{:}k{+}1{:}k{+}2$, it has to hold at most ¼ of the total energy only, and even less in some irregular modes. The excitation of the network, at the frequency $l\omega_0$, is not at one of the resonances of the network, and so the input current doesn't grow along with the output voltage, but falls back to zero when it reaches the maximum. Its envelope is limited at (exact limit for even l):

$$I_{\max} = \frac{C_a}{\sqrt{C_b L_b}} V_{in} \frac{k^2 m^2}{l^3 (m-k)} \tag{14}$$

The excitation away from the resonances is somewhat counterintuitive, but it actually produces the largest voltage gain within a limited number of cycles. And when the energy transfer is complete, there is no energy remaining in the network, except for the energy in C_b. The only problem is the need of the driver system to excite the network.

Example 2:

Consider a network designed for mode 18:19:20, with C_a = 5 nF, L_b = 28.2 mH, and C_b = 15 pF. The normalized transformerless network (fig. 4a) results in (from (11)):

C_1= 0.0111342593 F; L_1= 0.2501732502 H;
C_2= 1 F; L_2= 0.0027700831 H

And the final element values (fig. 4b), from (12), are:

C_a= 5 nF; L_a= 86.0126111111 uH;
C_b= 15 pF; L_b= 28.2 mH;
k_{ab}= 0.1046489272

The voltage gain (13) reaches 346.05, 18.95 times greater than the voltage gain of a conventional Tesla transformer. Some of the obtained waveforms for excitation with a 180 V cosinusoid are shown in fig. 5. The maximum voltage over C_a reaches 1701 V, the maximum input current is 13.03 A ((14) gives 13.07 A), the maximum current in L_b is 1.437 A, and the maximum output voltage is 62.29 kV.

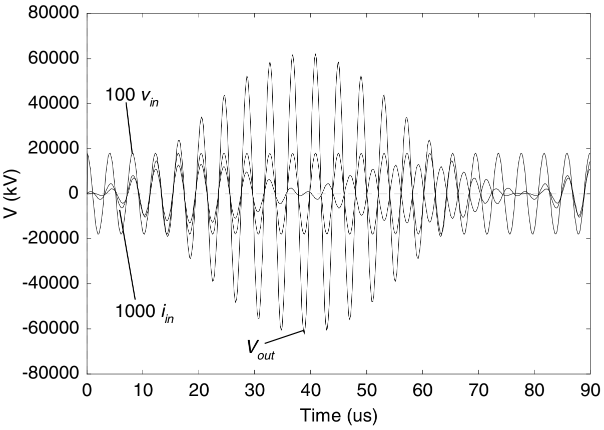

Figure 5. Voltages and input current for cosinoidal input.

With cosinusoidal input, the generation of networks with order higher than 4 is not practical, because always several terms of the expansion (5) disappear when $\varepsilon = \pm 1$, and the excitation of the network must be done with a sum of cosinusoids with different frequencies instead of with a single cosinusoid. The 4th-order network can also be excited with a bipolar square wave, with the first pulse having half of the normal width. The resulting waveforms are almost identical, as if the excitation were with a cosinusoid having an amplitude equal to $4/\pi$ times the amplitude of the square wave. Complete energy transfer can't be obtained, but the difference to the studied case is negligible when the energy transfer takes more than a few cycles.

IV. Incomplete Energy Transfer from Inductor to Capacitor

A more flexible design is obtained departing from the case of energy transfer from the input inductor L_1 to the output capacitor C_p (fig. 6).

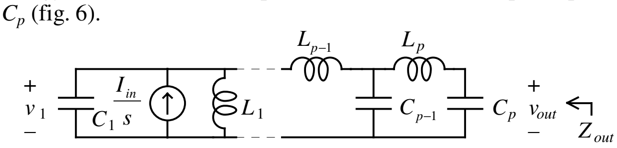

Figure 6. Energy transfer from L_1 to C_p.

An input current in L_1 causes an output voltage with the form (15), which can be represented as a sum of sinusoid transforms (16):

$$V_{out}(s) = \frac{\alpha}{\left(s^2 + k_1^2\omega_0^2\right)\left(s^2 + k_2^2\omega_0^2\right)\cdots\left(s^2 + k_p^2\omega_0^2\right)} \tag{15}$$

$$V_{out}(s) = \sum_{j=1}^{p} \frac{B_j k_j}{s^2 + k_j^2 \omega_0^2} \tag{16}$$

In this case the multipliers k_j must be all odd, with double odd differences, for proper adding of the sinusoidal components with maximum amplitude at $t = \pi/(2\omega_0)$. The output impedance is obtained as in the first case (3), but using the residues B_j (16) instead of the A_j [4]. Incomplete energy transfer in this case can be obtained by specifying that some voltage shall remain in C_1 at the end of the energy transfer (it's impossible to leave current in L_1 when v_{out} is maximum). Changing the origin of time to $t = \pi/(2\omega_0)$, the output voltage becomes a sum of cosinusoids caused by initial voltages on C_1 and C_p. The effect of the voltage in C_1 can be computed by (1) and (2), and we have a situation similar to the first case:

$$V_{out}(s) = \sum_{j=1}^{p} \frac{|B_j| s}{s^2 + k_j^2 \omega_0^2} = Z_{out}(s) + \varepsilon \sum_{j=1}^{p} \frac{A_j s}{s^2 + k_j^2 \omega_0^2} \tag{17}$$

And so $Z_{out}(s)$ can be directly obtained as (18) and the network obtained exactly as before.

$$Z_{out}(s) = \sum_{j=1}^{p} \frac{\left(|B_j| - \varepsilon A_j\right) s}{s^2 + k_j^2 \omega_0^2} \tag{18}$$

There are again two extreme values for ε that produce realizable networks, but now they are different from ± 1 because the A_j are different from the B_j. The two limits are the ratios $|B_j|/A_j$ with smaller magnitude, that generally correspond to the two highest frequencies in the waveforms. It's not clear if these networks have some practical interest, but the synthesis procedure is the base of the much more interesting networks described in the next section.

V. Operation with Zero-State Response - Sinusoidal Input

When ε assumes its two extreme values, exactly one term of the expansion (18) disappears, leaving an LC ladder network driven by a sinusoidal voltage source. And this happens for networks of any even order. In all cases two different networks can be generated, one for each of the extremes of ε. The low-pass case produces only transformerless networks, as in the cosinusoidal input case, and is of limited interest. Band-pass networks are more interesting, and can be generated by simply replacing α by αs^2 in (15) [6]. For mode k:l:m and ω_0 = 1 rad/s, the 6th-order case results in:

$$
\begin{aligned}
&V_{out}(s)=\frac{\alpha s^2}{(s^2+k^2)(s^2+l^2)(s^2+m^2)}=\frac{B_1 k}{s^2+k^2}+\frac{B_2 l}{s^2+l^2}+\frac{B_3 m}{s^2+m^2};\\
&B_1=\frac{k(m-l)}{(k+l)(k-m)};\; B_2=\frac{l(k+m)}{(k+l)(l+m)};\; B_3=-\frac{m(k-l)}{(k-m)(l+m)};\\
&\varepsilon_1=\frac{|B_1|}{A_1}=\frac{-2l^2(k+m)}{k(k+l)(l+m)};\\
&\varepsilon_2=\frac{|B_2|}{A_2}=\frac{2l(k+m)}{(k+l)(l+m)};\; \varepsilon_3=\frac{|B_3|}{A_3}=\frac{-2l^2(k+m)}{m(k+l)(l+m)};
\end{aligned}
\tag{19}
$$

The A_j are obtained as in (11). The limits of ε are ε_2 and ε_3, since ε_1 always produce a negative residue in $Z_{out}(s)$. Using ε_2, the excitation is at the central frequency $l\omega_0$. $Z_{out}(s)$ and the elements values for fig. 4a are:

$$
\begin{aligned}
&Z_{out2}(s)=\frac{(-B_1-\varepsilon_2 A_1)s}{s^2+k^2}+\frac{(-B_3-\varepsilon_2 A_3)s}{s^2+m^2}=\frac{s^3+((k^2m-km(l-m))/l)s}{(s^2+k^2)(s^2+m^2)};\\
&C_2=1;\; L_2=\frac{k-l+m}{klm};\\
&L_1=\frac{l(k-l+m)}{(k-l)(k+m)^2(l-m)};\; C_1=\frac{(l-m)(k+m)^2(k-l)}{km(k-l+m)^2}
\end{aligned}
\tag{20}
$$

Using ε_3, the excitation is at the upper frequency $m\omega_0$, and the impedance and element values are:

$$
\begin{aligned}
&Z_{out2}(s)=\frac{(-B_1-\varepsilon_3 A_1)s}{s^2+k^2}+\frac{(B_2-\varepsilon_3 A_2)s}{s^2+l^2}=\frac{s^3+((k^2l-kl(l-m))/m)s}{(s^2+k^2)(s^2+l^2)};\\
&C_2=1;\; L_2=\frac{k-l+m}{klm};\\
&L_1=\frac{-m(k-l+m)}{(k-l)^2(k+m)(l-m)};\; C_1=\frac{-(l-m)(k+m)(k-l)^2}{kl(k-l+m)^2}
\end{aligned}
\tag{21}
$$

The design (20), when converted to the form of fig. 4b through (12), results in more practical values and the highest voltage gain for the number of cycles of energy transfer (that is l/4 for this case and l/2 for cosinusoidal input):

$$A_v=\sqrt{\frac{C_a}{C_b}}\sqrt{\frac{km}{(l-m)(k-l)}} \tag{22}$$

The design (21) results in a much lower gain:

$$A_v=\sqrt{\frac{C_a}{C_b}}\sqrt{\frac{kl}{(k+m)(l-m)}} \tag{23}$$

Example 3:

A design using (20) that is close to the design in example 2 can be obtained with mode 17.5:18.5:19.5 (could be also 35:37:39). Again with C_a = 5 nF, L_b = 28.2 mH, and C_b = 15 pF, the normalized transformerless network (fig. 4a) is:

C_1= 0.0117216117F; L_1= 0.25 H;
C_2= 1 F; L_2= 0.0029304029 H

And the final element values (fig. 4b), from (12), are:

C_a= 5 nF; L_a= 85.5916483516 uH;
C_b= 15 pF; L_b= 28.2 mH;
k_{ab}= 0.1076373884

The difference from the previous design is small in these high modes. The voltage gain (23) reaches 337.27, smaller because the mode was moved to the next below. The same waveforms for excitation with a 180 V sinusoid are shown in fig. 7. The maximum voltage over C_a reaches 1661 V, the maximum input current is 12.77 A, the maximum current in L_b is 1.401 A, and the maximum output voltage is 60.71 kV.

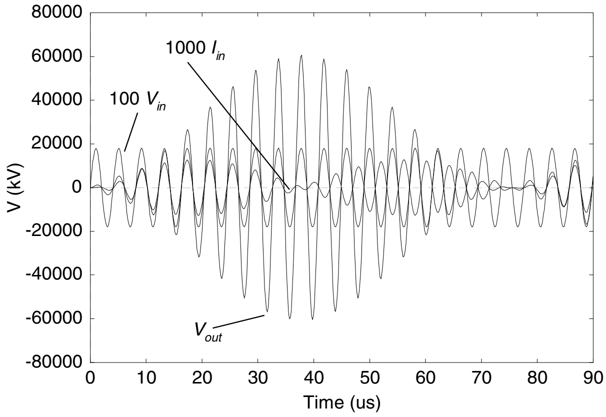

Figure 7. Voltages and input current for sinusoidal input.

VI. CONCLUSIONS

Two different procedures for the design of multiple resonance networks with incomplete energy transfer were described, and each of them lead to the design of classes of these networks operating with the zero-state response instead of with the zero-input response. These networks can be used instead of the classical Tesla transformer in applications where a capacitive load must be charged to high voltage, with energy coming from a low-voltage supply. The examples in the paper covered in detail only some particular cases, but the idea can be generalized to all the types of networks discussed in the references.

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