SYNTHESIS OF MULTIPLE RESONANCE NETWORKS

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ABSTRACT

This paper shows how “multiple resonance networks” of any order an with several structures can be systematically designed using standard passive filter design techniques. These networks are composed of capacitors, inductors, transformers, and a switch, and have the property of transferring completely energy initially put in one capacitor, insulated by the switch, to another capacitor in the circuit through a linear transient when the switch is closed, usually with the objective of producing very high voltages.

1. INTRODUCTION

“Multiple resonance networks” is a name that generalizes the “double resonance” [3][4], “triple resonance” [5][6], and the higher-order versions discussed in this paper. These circuits are usually composed of a transformer and some extra capacitors and inductors, and work by transferring the energy initially stored in a capacitor at one side of the transformer to another, much smaller, capacitor at the other side of the transformer, through a linear transient composed (in the ideal lossless case) of a sum of several sinusoidal waveforms (Fig. 1).

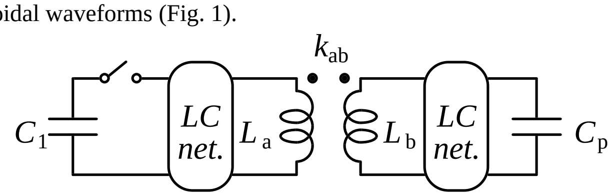

Figure 1. Multiple resonance network. An initial energy in C_1 is totally transferred to C_p through the transformer and two possible *LC* networks, during the transient after the closing of the switch.

The double resonance case is long known and found applications ranging from early radio systems [2] and electrotherapeutic devices [1], to the generation of high voltages in high-energy physics instrumentation [5] and the production of impressive sparks for demonstrations about electricity [4] (the “Tesla coil”). Only the two capacitors and the transformer are used, what results in a 4th-order system with a transient formed by two oscillatory modes (Fig. 2). The analysis and design equations for this circuit can be found in many papers [3][4] and books. With the system properly designed, and small losses, the voltage in C_1 decays cosinusoidally at each cycle while the voltage in C_2 rises sinusoidally. When the voltage in C_2 reaches the maximum, the voltage in C_1 is zero, and the currents at both sides of the transformer are also null. In the ideal lossless case, all the original energy goes to C_2, and the obtained voltage is given, by simple conservation of energy, by eq. 1 (with C_p=C_2). Note that this equation fixes the maximum output voltage for a system of this type, independently of the order or of the structure, and is valid for the other systems discussed below.

$$V_{out} = V_{in}\sqrt{\frac{C_1}{C_p}} \tag{1}$$

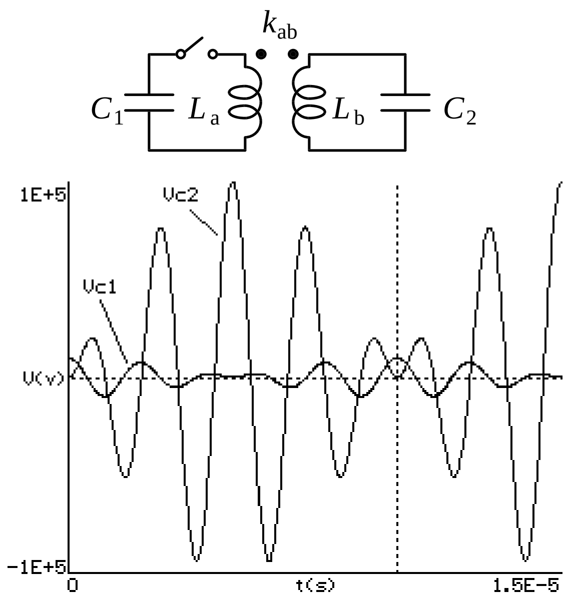

Figure. 2. Typical double resonance network, and typical voltages in C_1 and C_2 after the closing of the switch. The voltage gain was designed as 10. C_1=1 nF, C_2=10 pF, L_a=129.8 μH, L_b=12.98 mH, k_{ab}=0.2195. $VC_1(0)$=10 kV.

Several variations of this circuit that can be found in the literature, as using an autotransformer (“Oudin coil”), with extra inductors at one or both sides [2], or using no transformer, but a “T” or “L” (as in this paper) of inductors. Equivalencies can be found among all the versions.

More recently, triple resonance systems were developed [5][6][7] for instrumentation used in high-energy physics. In the studied case, an additional capacitor and an inductor are added to the output side (fig. 3), with the aim of reducing the voltage stress over the transformer and of taking into consideration the output capacitance of the transformer. With only the extra inductor added, the system is still a double resonance system, long known too [2], the “Tesla magnifier”. With the extra capacitor the system is now of 6th order, the transient has three oscillatory modes, but the operation with complete energy transfer is equally possible. Design formulas can be found in [5][6][7], and in [8] (with some corrections).

The system can be used also with the additional inductor and capacitor at the input side, with the aim of reducing the energy dissipation in the resistance of the switch. This idea can be found in [2]. The operation is identical, using the “return part” of the transient, where in the circuit in fig. 3 the energy returns to C_1.

In all the cases found in the literature the design of these systems is based on the analysis of a fixed structure. The following section shown that the design can be made by synthesis, can be applied to a wide range of structures, and can be extended to systems of any order.

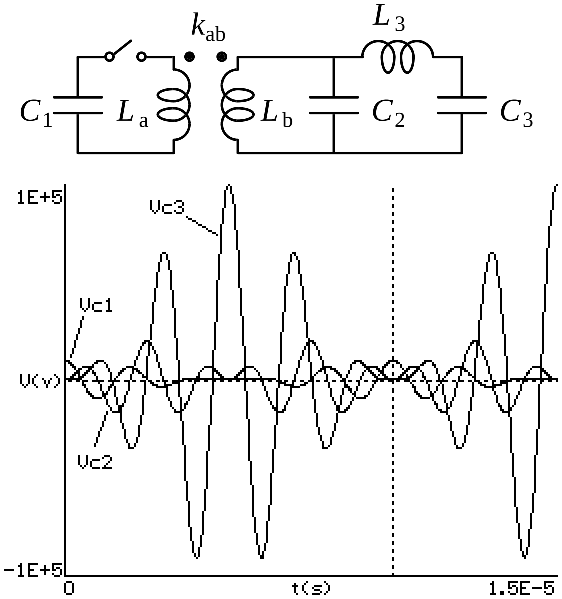

Figure 3. Typical triple resonance network, and typical voltages in C_1, C_2, and C_3 after the closing of the switch. The currents in the transformer an at the extra inductor are also all null when the output voltage is maximum. The voltage gain was also designed to 10. C_1=1 nF, C_2=126.3 pF, C_3=10 pF, L_a=110 μH, L_b=780.7 μH, L_3=10.13 mH, k_{ab}=0.28131, $VC_1(0)$=10 kV.

2. SYNTHESIS APPROACH

A natural form for the *LC* networks in fig. 1, extending the structure in fig. 3, is a ladder sequence of series inductors and shunt capacitors. The transformer can be left out of the problem, because it can be eliminated, or inserted after the synthesis, using the equivalence shown in fig. 4a, where:

$$\begin{aligned} L_a &= \frac{L_x}{n^2} \\ L_b &= L_x + L_y \\ k_{ab} &= \sqrt{\frac{L_x}{L_x + L_y}} \\ Z' &= \frac{Z}{n^2} \end{aligned} \tag{2}$$

n is the turns ratio for the transformer, that can be chosen as convenient for the desired voltage gain. It multiplies the voltage gain directly.

This reduces the problem to the synthesis of an *LC* network with the structure of an *LC* odd-order ladder polynomial low-pass filter, with an added shunt inductor somewhere. If the inductor is at the low-voltage end (as is the case for the 4th and 6th order cases in figs. 2 and 3, with the transformer eliminated, and also for the trivial 2nd order case), the problem is reduced to the synthesis of the output impedance of the circuit by a succession of complete pole removals at infinity, or a synthesis in Cauer's first form.

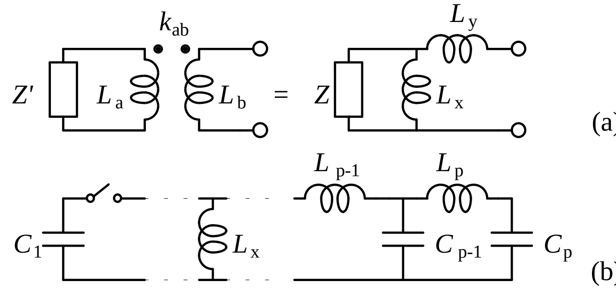

Figure 4. Equivalence that allows the insertion of a transformer where the shunt inductor appears (a), and general structure without transformer (b).

With the switch closed, the impedance seen from the input side of the network has a denominator of order *p*, even, and a numerator of order *p*–1, with a zero at DC. The same is valid for the impedance seen at the output side. The voltage response of one of these impedances to a current impulse applied in parallel with the corresponding input or output capacitor, that is proportional to the response to a charged capacitor there, is obtained from the inverse Laplace transform of the impedance seen at that side, and appears as a sum of *p* pure cosinusoidal oscillations with positive multiplying factors. Sinusoidal components don't appear and the multiplying factors must be positive because the Laplace transform of the resulting waveform is proportional to the impedance at that side in Foster's first form:

$$Z_{in,i}(s) \propto V_{Ci}(s) = \sum_{j=1}^{p} \frac{A_{ij}\, s}{s^2 + k_j{}^2 \omega_0{}^2} \tag{3}$$

Similar considerations apply also to the internal capacitors in the structure, and so all the capacitor voltages have the forms shown in eq. 3 (Laplace transform) and eq. 4 (time domain), where the *p* oscillation frequencies are considered as multiples of a basic frequency ω_0:

$$V_i(t) = \sum_{j=1}^{p} A_{ij} \cos\left(k_j \omega_0 t\right) \tag{4}$$

The currents at all the inductors are then proportional to the derivatives of the capacitor voltages (eq. 4), and so are all sums of *p* sinusoids at the same frequencies:

$$I_i(t) = \sum_{j=1}^{p} B_{ij} \sin\left(k_j \omega_0 t\right) \tag{5}$$

For convenience, let's work from the output of the network, and compute the output impedance. This impedance appears simply in Cauer's first form if the transformer (if later included) is to be directly connected to the first capacitor, C_1. Considering C_p initially charged to V_p, the energy there is transferred to C_1 using the "return" part of the (perpetual) transient waveform. Considering that the first instant where complete energy transfer occurs is $t = \pi/\omega_0$, it is evident that the condition that causes all the currents (eq. 5) to be zero at this instant is that the k_j must be different positive integers.

With the coefficients A_{pj} in eq. 4 computed, the entire network can be obtained simply by the expansion of the impedance in eq. 3, conveniently scaled. A convenient scaling is to consider the output excited by an unitary impulsive source, what transforms the proportionality in eq. 3 into an identity. The conditions for the complete energy transfer are:
From eq. 3 and fig. 4b, when $s \to \infty$:

$$\sum_{j=1}^{p} A_{pj} = \frac{1}{C_p} \tag{6}$$

At $t = \pi/\omega_0$, $V_p=0$. From eq. 4:

$$V_p(t) = \sum_{j=1}^{p} A_{pj} \cos(k_j \pi) = \sum_{j=1}^{p} A_{pj} (-1)^{k_j} = 0 \tag{7}$$

At the same instant, $V_{p-1}=0$, $V_{p-2}=0$, …, $V_2=0$. At any time (except t = 0 to remove the input current impulse):

$$V_{p-1}(t) = V_p(t) + L_p \frac{dI_p(t)}{dt} = V_p(t) + L_p C_p \frac{d^2 V_p(t)}{dt^2} \tag{8}$$

$$V_{p-2}(t) = V_{p-1}(t) + L_{p-1} \frac{dI_{p-1}(t)}{dt} = V_{p-1}(t) + L_{p-1} C_{p-1} \frac{d^2 V_{p-1}(t)}{dt^2} \tag{9}$$

$$V_2(t) = V_3(t) + L_3 \frac{dI_3(t)}{dt} = V_3(t) + L_3 C_3 \frac{d^2 V_3(t)}{dt^2} \tag{10}$$

These voltages are all zero at $t = \pi/\omega_0$, if all the even derivatives of $V_p(t)$ up to order $2p - 4$ are null at this instant. Combining this condition with eqs. 6 and 7, and eliminating powers of ω_0 and of -1 that multiply the equations the following system of equations results:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ (-1)^{k_1} & (-1)^{k_2} & \cdots & (-1)^{k_p} \\ k_1^2(-1)^{k_1} & k_2^2(-1)^{k_2} & \cdots & k_p^2(-1)^{k_p} \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{2p-4}(-1)^{k_1} & k_2^{2p-4}(-1)^{k_2} & \cdots & k_p^{2p-4}(-1)^{k_p} \end{bmatrix} \begin{bmatrix} A_{p1} \\ A_{p2} \\ \vdots \\ \vdots \\ A_{pp} \end{bmatrix} = \begin{bmatrix} 1/C_p \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{11}$$

As all the A_{pj} are positive and the even powers of the k_j are also positive (even if some of the k_i were negative), the powers of -1 must have different signs in this system of equations, so satisty the sums that result in zero. This adds a condition on the k_j, mentioned in [3][5][6][7], and that extends for higher orders: Given a positive integer as k_j, the next value $k_{j+1}$is obtained by adding an odd integer to k_j. Valid sequences are 1, 2, 3, …; 2, 3, 4, …; 1, 2, 5, …; 1, 4, 5, …; etc. With this condition, the system (11) is reduced to:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & (-1)^{p-1} \\ k_1^2 & -k_2^2 & k_3^2 & \cdots & k_p^2(-1)^{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_1^{2p-4} & -k_2^{2p-4} & k_3^{2p-4} & \cdots & k_p^{2p-4}(-1)^{p-1} \end{bmatrix} \begin{bmatrix} A_{p1} \\ A_{p2} \\ A_{p3} \\ \vdots \\ A_{pp} \end{bmatrix} = \begin{bmatrix} 1/C_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{12}$$

With the A_{pj} computed, an LC network can be obtained by the expansion of the output impedance (eq. 3). Note that any structure that puts C_1 "at the maximum distance" from C_p is valid. The simplest structure is shown in fig. 4b, but other possibilities exist.

3. EXAMPLES

3.1 4th-order case:

This is the classic double resonance circuit, but without a transformer (that can be included as discussed above). The system of equations in eq. 12, with C_2 normalized to 1 reduces to two equations that give $A_{21} = A_{22} = 1/2$. The output impedance of the network is then, normalizing ω_0 to 1 and naming $k_1=k$ and $k_2=l$:

$$Z_{out} = \frac{\frac{1}{2}s}{s^2+k^2} + \frac{\frac{1}{2}s}{s^2+l^2} = \frac{s^3 + \frac{1}{2}\left(k^2+l^2\right)s}{s^4 + \left(k^2+l^2\right)s^2 + k^2 l^2} \tag{13}$$

This impedance, expanded in Cauer's first form (and rearranged as in fig. 4b) results in the structure in fig. 5a, with the component values:

$$C_2 = 1; \qquad L_2 = \frac{2}{k^2+l^2};$$
$$C_1 = \left(\frac{k^2+l^2}{l^2-k^2}\right)^2; \; L_1 = \frac{\left(l^2-k^2\right)^2}{2\left(k^2+l^2\right)k^2 l^2} \tag{14}$$

Note that even without a transformer this circuit produces the voltage gain (from eq. (1)):

$$\frac{V_{C2\max}}{V_{C1}} = \sqrt{\frac{C_1}{C_2}} = \frac{k^2+l^2}{l^2-k^2} \tag{15}$$

3.2 6th-order case:

For the triple resonance case, the system (eq. 12) has 3 equations. Symbolic expressions for the element values can be obtained and are listed in eqs. 16, for the structure in fig. 5b, with $k_1=k$, $k_2=l$, and $k_3=m$.

$$A_{31} = \frac{l^2-m^2}{2\left(k^2-m^2\right)}; \; A_{32} = \frac{1}{2}; \; A_{33} = \frac{k^2-l^2}{2\left(k^2-m^2\right)};$$
$$C_3 = 1; L_3 = \frac{1}{l^2};$$
$$C_2 = \frac{2l^4}{\left(l^2-m^2\right)\left(k^2-l^2\right)};$$
$$L_2 = \frac{\left(l^2-m^2\right)\left(k^2-l^2\right)}{l^2\left(k^2\left(l^2+m^2\right)-l^2\left(l^2-m^2\right)\right)}; \tag{16}$$
$$C_1 = \frac{\left(k^2\left(l^2+m^2\right)-l^2\left(l^2-m^2\right)\right)^2}{\left(k^2-l^2\right)^2\left(l^2-m^2\right)^2};$$
$$L_1 = \frac{\left(k^2-l^2\right)^2\left(l^2-m^2\right)^2}{2k^2 l^2 m^2\left(k^2\left(l^2+m^2\right)-l^2\left(l^2-m^2\right)\right)}$$

And the voltage gain is given by:

$$\frac{V_{C3\max}}{V_{C1}} = \sqrt{\frac{C_1}{C_3}} = \frac{k^2\left(l^2+m^2\right)-l^2\left(l^2-m^2\right)}{\left(k^2-l^2\right)\left(l^2-m^2\right)} \tag{17}$$

These expressions reduce to the expressions in [6][8] if a transformer is included. Note that there is another possibility, with the shunt inductor in parallel with C_2, at the center of the structure. The resulting structure is symmetrical and has no voltage gain, but with the inclusion of a transformer both versions result in the same circuit.

3.3 8th order case:

The extension for higher orders results in more voltage gain for the same (lower) frequencies of operation, and smaller voltage differences across the series inductors. No attempts of practical applications of networks with orders greater than 6 could be found in the literature. Symbolical expressions continue to be relatively easy to derive for the component values, but become rather large for the "quadruple resonance" case and above. Table 1 lists numerical normalized (C_p=1, ω_0=1) element values for the structure in fig. 5c for some of the possibly more practical combinations of frequency multipliers. Fig. 6 shows the voltage waveforms for the 2, 3, 4, 5 mode.

Table 1: Normalized element values for quadruple resonance networks (fig. 5c), as function of the frequency multipliers k_1, k_2, k_3, and k_4. In all cases the total energy transfer occurs in π seconds.

Values/Mode	*1, 2, 3, 4*	*2, 3, 4, 5*	*3, 4, 5, 6*
C_4	1.000000	1.000000	1.000000
L_4	0.181818	0.086957	0.051282
C_3	1.920635	3.918519	6.584416
L_3	0.078456	0.020310	0.007395
C_2	3.383492	12.626002	34.290043
L_2	0.051554	0.006828	0.001491
C_1	7.060408	50.342404	227.082720
L_1	0.051453	0.002312	0.000266
Voltage gain	2.657143	7.095238	15.069264

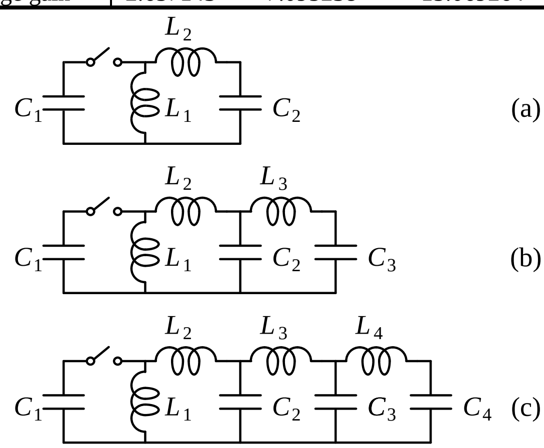

Figure 5. Structures for transformerless voltage multipliers of 4th (a), 6th (b), and 8th (c) orders.

4. CONCLUSIONS

A systematic procedure for the design of *LC* voltage multipliers that are a generalization of the "double resonance" and "triple resonance" networks was presented. The networks can be designed in several ways, and in the most useful form, the one that produces the largest voltage gain, by the direct expansion of an *LC* impedance in Cauer's first form (continuous fraction expansion). It was also concluded that it is not necessary, if only a voltage gain is required, to use a transformer, and that a transformer can easily be included in the ladder networks designed by the proposed method by a simple circuit transformation. Only lossless circuits were considered, but the applications of these circuits usually require low losses, and they are designed to behave as lossless circuits. The presence of small losses don't affect significantly the behavior of practical circuits of this type. All the proposed networks are of even order, but note that odd-order networks, without the shunt inductor, are also possible through simple modifications in the synthesis procedure, by the addition of a pole at 0 in eq. 3. These networks, however, result always symmetrical, show no input-output voltage gain, and no place where a transformer can be included.

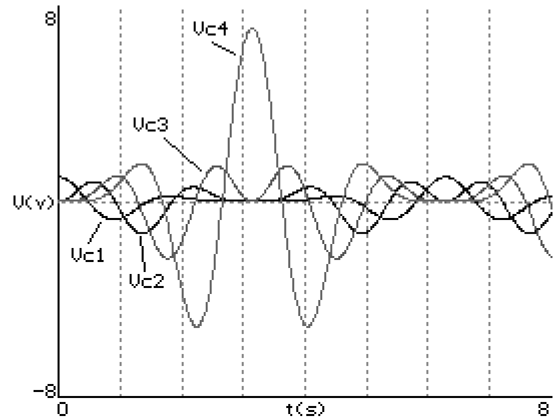

Figure 6. Voltage waveforms for the normalized 8th-order network in fig. 5. The voltage gain of 7.095 can be observed, for an unitary starting voltage in C_1.

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