Mutual Inductance and Inductance Calculations by Maxwell's Method

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Abstract—The classical book by James Clerk Maxwell, "A Treatise on Electricity and Magnetism" (1873) [1], described an interesting method for the calculation of inductances, derived from a method that calculates mutual inductances. The method was implemented in the program **Inca**, available at *http://www.coe.ufrj.br/~acmq /programs*. This article discusses the implementation, and also discusses several other formulas for inductance and mutual inductance calculation.

I. Mutual inductance

The mutual inductance between two current filaments can be calculated by Neumann's formula:

$$M=\frac{\mu_0}{4\pi}\iint\frac{\vec{ds}\cdot\vec{ds}'}{r} \tag{1}$$

where ds and ds'are incremental sections of the filaments, the dot means scalar product, and r is the distance between them. The exact integral is obtained from an adequate parametrization of the geometry of the filaments.

701.]† The mutual inductance between two coaxial filamental circles, one with radius a and another with radius A, with distance between centers b, can be calculated as:

$$\begin{aligned}
&M_{12}=\frac{\mu_0}{4\pi}\iint\frac{\cos\varepsilon}{r}\,ds\,ds';\\
&r=\sqrt{A^2+a^2+b^2-2Aa\cos(\varphi-\varphi')};\\
&\varepsilon=\varphi-\varphi';\\
&ds=a\,d\varphi;\\
&ds'=A\,d\varphi';\\
&M_{12}=\frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{Aa\cos(\varphi-\varphi')d\varphi d\varphi'}{\sqrt{A^2+a^2+b^2-2Aa\cos(\varphi-\varphi')}}
\end{aligned} \tag{2}$$

This integral can be exactly solved in the form:

$$\begin{aligned}
&M_{12}=-\mu_0\sqrt{Aa}\left[\left(k-\frac{2}{k}\right)K+\frac{2}{k}E\right];\\
&k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+b^2}}
\end{aligned} \tag{3}$$

where K and E are the complete elliptic integrals of first and second kinds with modulus k:

$$\begin{aligned}
&K=\mathrm{F}(k,\pi/2)=\mathrm{F}(k)=\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}\\
&E=\mathrm{E}(k,\pi/2)=\mathrm{E}(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\,d\varphi
\end{aligned} \tag{4}$$

To calculate the mutual inductance between two concentrical coils with integer number of turns, the coils 1 and 2 are first decomposed on sets of n_1 and n_2 circular closed loops, and the total mutual inductance is obtained from the evaluation of:

$$M_{total}=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}M_{ij} \tag{5}$$

where M_{ij} is the mutual inductance between the loops i and j. (It's possible to have one of the coils with the last turn incomplete. (3) gives the right answer when one of the turns covers just θ radians if multiplied by θ/(2π).)

II. Self-inductance

693.]† The inductance of a coil with uniform section, where the radius of curvature is large compared with the dimensions of the transverse section of the conductor, can be calculated by computing the mutual inductance between two filamental conductors placed at a distance equal to the geometrical mean distance or every pair of points in the section of the conductor. The geometrical mean distances for a round conductor with radius r, for a flat wire of width a, and for a square wire with side b are:

$$\begin{aligned}
&R=r\,e^{-\frac{1}{4}}=0.7788\,r\\
&R=ae^{-\frac{3}{2}}=0.22313\,a\\
&R=e^{\log b+\frac{1}{3}\log 2+\frac{\pi}{3}-\frac{25}{12}}=0.44705\,b
\end{aligned} \tag{6}$$

The calculation in this way assumes uniform current in the wire.

Inductance of a solenoid

In the case of a solenoid with integer number of turns, the double sum (5) can be greatly simplified, because there are only $2n$-1 different terms to compute, instead of the n^2 of the general case. Considering the turn numbers as i in one coil and i' in the other, placed vertically at a distance R, the mutual inductance between turn 1 and turn 1', $M_{11'}$, appears n

† Numeration in Maxwell's book.

Created: 14 March 2003. Last update: 20 October 2014

times, $M_{21'}$ and $M_{12'}$ appear n-1 times, $M_{31'}$ and $M_{13'}$ appear n-2 times, and so on, until $M_{n1'}$ and $M_{1n'}$ that appear just 1 time. If the image coil were assembled outside or inside, instead of above, just n different terms would be necessary, but the coils would be different, and the error probably larger. See Kirchhoff's formula below for a similar approach.

The Pascal routine used in Inca (with the drawing routines and messages removed) is shown below:

```
{
Inductance of a solenoid by Maxwell's
method, using elliptic integrals
Rounds the number of turns, n≥1
}
function MaxwellLEl(n,h,r,b,d:real):real;
var
  a1,c,b1b2,RM,z1,z2,z10,soma,turn1,
  turn2:real;
  v,vt:integer;
begin
  vt:=round(n);
  RM:=d/2*exp(-0.25); {g. m. d.}
  a1:=h/vt;
  b1b2:=RM;
  z10:=b+a1/2;
  z1:=z10;
  z2:=z10;
  for v:=1 to vt do begin
    c:=2*r/sqrt(sqr(2*r)+sqr(z1-z2-b1b2));
    EF(c);
    turn1:=-r*((c-2/c)*Fk+(2/c)*Ek);
    if v=1 then soma:=vt*turn1
    else begin
      c:=2*r/sqrt(sqr(2*r)+sqr(z1-z2+b1b2));
      EF(c);
      turn2:=-r*((c-2/c)*Fk+(2/c)*Ek);
      soma:=soma+(vt-(v-1))*(turn1+turn2);
    end;
    z1:=z1+a1;
  end;
  MaxwellLEl:=4e-7*pi*soma;
end;
```

Flat and conical coils

A conical of flat coil doesn't admit this simplification, but can still be decomposed in a series of circular rings. The mutual inductance between two coaxial conical coils can be still calculated by (5), and the self inductance can be calculated as the mutual inductance between two identical coils separated by (6).

Evaluation of the elliptic integrals

The complete elliptic integrals can, in principle, be evaluated by the series:

$$K = \frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right\}$$
$$E = \frac{\pi}{2}\left\{1-\left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right\} \quad (7)$$

A problem is that if R is much smaller than the radius of the loops, the modulus k in (3) tends to 1 in the integrals involving a turn and its adjacent copy, and the evaluation of K becomes problematic. The series converges very slowly, and easily millions of terms must be used. Numerical integration is an alternative when this happens, but it must be performed with high resolution due to the large derivatives of the integrand close to the end of the interval (about 100000 intervals with an uniform Simpson's rule are necessary for good precision up to k = 0.999999999). It is possible to use different series, that converge quickly for k close to 1 [15]. However, a very simple algorithm exists, the AGM (arithmetic-geometric mean) method, the produces accurate values quickly. A Pascal function that evaluates $F(c)$ and $E(c)$ using the AGM method, implemented in the Inca program, is:

```
{
Complete elliptic integrals of first
and second classes - AGM method.
Returns the global variables:
Ek=E(c) and Fk=F(c)
Doesn't require more than 7 iterations for
c between 0 and 0.9999999999.
Reference: Pi and the AGM, J. Borwein and
P. Borwein, John Wiley & Sons.
}
procedure EF(c:real);
var
  a,b,a1,b1,E,i:real;
begin
  a:=1;
  b:=sqrt(1-sqr(c));
  E:=1-sqr(c)/2;
  i:=1;
  repeat
    a1:=(a+b)/2;
    b1:=sqrt(a*b);
    E:=E-i*sqr((a-b)/2);
    i:=2*i;
    a:=a1;
    b:=b1;
  until abs(a-b)<1e-15;
  Fk:=pi/(2*a);
  Ek:=E*Fk
end;
```

True spiral coils

The equation that gives the mutual inductance between two general coaxial conical coils is a more general version of (1) (see fig. 1):

$$M_{12} = \frac{\mu_0}{4\pi}\int_0^{2\pi n_1}\int_0^{2\pi n_2} \frac{dx_1 dx_2 + dy_1 dy_2 + dz_1 dz_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}} \quad (8)$$

where, for i=1 and 2:

$$\begin{aligned}
&g_i = \frac{s_i - r_i}{2\pi n_i};\quad a_i = \frac{h_i}{2\pi n_i};\\
&x_i = (r_i + g_i\theta_i)\cos\theta_i;\quad y_i = (r_i + g_i\theta_i)\sin\theta_i;\\
&z_i = a_i\theta_i + b_i;\\
&dx_i = \left[-y_i + g_i\cos\theta_i\right]d\theta_i;\quad dy_i = \left[x_i + g_i\sin\theta_i\right]d\theta_i;\\
&dz_i = a_i d\theta_i
\end{aligned} \quad (9)$$

It is assumed that both spirals start at the same angle. This apparently irreducible integral [7] can be solved numerically. The self-inductance of a conical coil can be calculated by considering two identical coils separated by a vertical distance R', that is R with a small correction for the inclination of the wire:

$$R' = R\frac{\sqrt{(h/n)^2 + (2\pi r)^2}}{2\pi r} \tag{10}$$

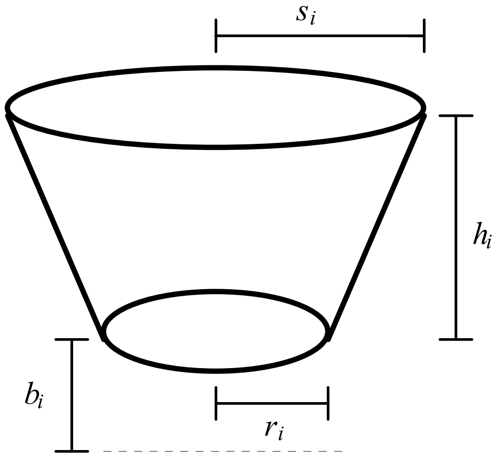

Fig. 1. Conical coil.

where h is the height of the coil, r is the radius of the turns (always measured between wire centers), and n is the number of turns. For a conical coil, the geometrical average of the radii is used, and the correction is approximate (the distance between wires varies along two stacked identical conical coils). The numerical integration must be done with high resolution, due to the small distance between the filaments.

True solenoidal coils

For solenoidal coils, considering two coaxial solenoids with radii r_1 and r_2, numbers of turns n_1 and n_2, heights h_1 and h_2, and base heights b_1 and b_2 , (8) becomes:

$$M = \frac{\mu_0}{4\pi}\int_0^{2\pi n_1}\int_0^{2\pi n_2}\frac{(r_1 r_2\cos(\varphi-\varphi') + a_1 a_2)d\varphi d\varphi'}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2\cos(\varphi-\varphi') + (a_1\varphi - a_2\varphi' + b_1 - b_2)^2}} \tag{11}$$

where $a_1 = h_1/(2\pi n_1)$ and $a_2 = h_2/(2\pi n_2)$.

For the self-inductance calculation, the program uses two identical coils separated vertically by R' (10). The same simplification of the case with circular windings appears, with only $2n$ integrations over single turns being necessary for the evaluation of the integral.

III. Explicit formulas for inductance

The Inca program also implements several formulas reported in the literature for the calculation of inductances of solenoids. In all cases, the formulas were adapted for inductances in Henrys and dimensions in meters.

Wheeler's approximate formula [2], for solenoids. Works well when the turns are closely spaced, giving a result similar to Lorenz' formula (14). A version of it for dimensions in meters is:

$$L = \mu_0\frac{\pi r^2 n^2}{h + 0.9r} \tag{12}$$

Wheeler's formula for flat coils [2] can be put in the form:

$$L = \frac{\mu_0}{4\pi}\frac{1000}{2.54}\frac{(r+s)^2 n^2}{60s - 28r} \tag{13}$$

Lorenz' formula [3], models a solenoid as a cylindrical current sheet, and works well for solenoids with thin windings of closely spaced turns. This equation is seen in several texts (see (16)) with slightly different equivalent forms:

$$\begin{aligned} L &= \mu_0\frac{8r^3}{3\varepsilon^2}\left[-1 + \frac{2k^2-1}{k^3}E + \frac{1-k^2}{k^3}K\right]; \\ k^2 &= \frac{4r^2}{h^2 + 4r^2};\ \ \varepsilon = \frac{h}{n} \end{aligned} \tag{14}$$

Kirchhoff's formula [4], decomposes the coil in circular turns, as done in Maxwell's method, and combines mutual inductances between turns calculated by elliptic integrals with $f(0)$, an approximation for the self-inductance of a single loop. α is the wire radius. For the case of a solenoid (the formula below), there is a simplification similar to the one described for Maxwell's method, with only n different mutual inductances that are calculated by Maxwell's formula:

$$\begin{aligned} &L = nf(0) + 2(n-1)f(\varepsilon) + 2(n-2)f(2\varepsilon) + \cdots + 2f((n-1)\varepsilon); \\ &f(z) = \mu_0\frac{r}{k}\left[(2-k^2)K - 2E\right]; \\ &k^2 = \frac{4r^2}{4r^2 + z^2}; \\ &f(0) = \mu_0 r\left(\mathrm{Ln}\frac{8r}{\alpha} - \frac{7}{4}\right);\ \ \varepsilon = \frac{h}{n} \end{aligned} \tag{15}$$

This formula can be easily adapted for coils of any shape that can be decomposed into coaxial circular rings. Russell [15] gives a derivation of $f(0)$, as an approximation for an expression using elliptic functions:

$$\begin{aligned} f(0) &= \mu_0 r\left(2(K-E) + \frac{1}{4}\right); \\ k &= \frac{r-\alpha}{r} \end{aligned} \tag{16}$$

Snow's formula [5][6] adds a complicated correction to Lorenz' formula. The result is similar to Maxwell's or Kirchhoff's formulas using circular turns, but the calculation is faster, without the summation. a is the coil radius, b is the coil height, and c is the wire diameter. The number of turns n shall be integer:

$$p=\frac{2a}{b};\ \ \theta=\tan^{-1}p;\ \ k=\sin\theta;\ \ k'=\cos\theta;\ \ z=\frac{\pi nc}{b};$$
$$\begin{aligned}L=\frac{\mu_0}{4\pi}\Bigg\{&\frac{8n^2a\pi}{3}\left[\frac{K+(p^2-1)E}{k}-p^2\right]+\\&+2\pi a\left[\begin{aligned}&2n\left(\frac{1}{4}-\ln z\right)+\frac{1}{3}\ln\left(\frac{2\pi na}{b}\right)-\frac{4}{\pi^2}\left(\frac{E}{k}-1\right)\left(1+\frac{z^2}{8}\right)-\\&-\frac{2}{3}\left(\frac{K-E}{k}-\frac{kK}{2}\right)-\frac{k'}{2k}\left(1-\frac{k'\theta}{k}\right)\end{aligned}\right]+\\&+b\left(\ln\frac{1+k'}{1-k'}+k'\ln 4\right)\Bigg\}\end{aligned}\tag{17}$$

IV. Explicit formulas for mutual inductance

An interesting solution involving true spiral coils was the formula for the mutual inductance between a circular loop and a true solenoid starting at it plane obtained by John Viriamu Jones [7]. A is the radius of the solenoid, a the radius of the loop, p the height of a turn divided by 2π, Θ is the final angle of the solenoid $2\pi n$, and $\prod(k,c)$ is the complete elliptic integral of the third kind. For solenoids at any distance from the loop, $M=M_{\Theta 2}-M_{\Theta 1}$. The paper also shows how to compute the mutual inductance between a cylindrical current sheet and a solenoid.

$$\begin{aligned}&M_\Theta=\frac{\mu_0}{4\pi}\Theta(A+a)ck\left[\frac{K-E}{k^2}+\frac{c'^2}{c^2}\left(K-\Pi(k,c)\right)\right];\\&c=\frac{2\sqrt{Aa}}{A+a};\ \ x=p\Theta;\ c'^2=1-c^2;\\&k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+x^2}}\end{aligned}\tag{18}$$

If c=1, the second term reduces to zero. The complete elliptic integral of the third kind:

$$\Pi(k,c)=\int_0^{\pi/2}\frac{d\varphi}{\left(1-c^2\sin^2\varphi\right)\sqrt{1-k^2\sin^2\varphi}}\tag{19}$$

can also be efficiently evaluated by an AGM algorithm [8]. Below is the Pascal routine used in the Inca program, that evaluates simultaneously the three elliptic integrals when they are needed. It requires at most 7 iteractions in the loop:

```
{
Complete elliptic integrals of first, second, and
third kinds - AGM
Returns the global variables Ek=E(k), Fk=F(k), and
IIkc=II(k,c)
Reference: Garrett, Journal of Applied Physics, 34,
9, 1963, p. 2571
}
procedure EFII(k,c:real);
var
  a,b,d,e,f,a1,b1,d1,e1,f1,S,i:real;
begin
  a:=1;
  b:=sqrt(1-sqr(k));
  d:=(1-sqr(c))/b;
  e:=sqr(c)/(1-sqr(c));
  f:=0;
  i:=1/2;
  S:=i*sqr(a-b);
  repeat
    a1:=(a+b)/2;
    b1:=sqrt(a*b);
    i:=2*i;
    S:=S+i*sqr(a1-b1);
    d1:=b1/(4*a1)*(2+d+1/d);
    e1:=(d*e+f)/(1+d);
    f1:=(e+f)/2;
    a:=a1;
    b:=b1;
    d:=d1;
    e:=e1;
    f:=f1;
  until (abs(a-b)<1e-15) and (abs(d-1)<1e-15);
  Fk:=pi/(2*a);
  Ek:=Fk-Fk*(sqr(k)+S)/2;
  IIkc:=Fk*f+Fk;
end;
```

With this formula, the mutual inductance between a coil with circular turns and a true solenoid can be easily calculated, by just adding all the mutual inductances between the individual turns and the solenoid.

The same formula can also be written as (adapting a formula in [8]):

$$\begin{aligned}&M_\Theta=\frac{\mu_0}{2}n\left\{z(K-E)+\frac{(A-a)^2}{z}\left[K-\Pi(k,c)\right]\right\};\\&z=\sqrt{(A+a)^2+x^2};\\&k=\frac{2\sqrt{Aa}}{z};\ \ c=\frac{2\sqrt{Aa}}{A+a}\end{aligned}\tag{20}$$

Again, the second term disappears if A=a. Another equivalent formula, that instead of the elliptic integral of the third kind uses incomplete elliptic integrals (the limits of the integrals in (4) are from 0 to θ) is [5]:

$$\begin{aligned}&M_\Theta=\frac{\mu_0}{2}\frac{n}{x}\left\{\begin{aligned}&\frac{2x\sqrt{Aa}}{k}(K-E)\pm\\&\pm\left|A^2-a^2\right|\left[KE(k',\theta)-(K-E)F(k',\theta)-\frac{\pi}{2}\right]\end{aligned}\right\};\\&k=\sqrt{\frac{4Aa}{x^2+(A+a)^2}};\ \ k'=\sqrt{1-k^2};\ \ \theta=sin^{-1}\sqrt{\frac{1+\left(\frac{x}{A+a}\right)^2}{1+\left(\frac{x}{A-a}\right)^2}}\end{aligned}\tag{21}$$

Curiously, the formula for the mutual inductance between a circular ring and a current sheet solenoid is identical to these formulas, that consider a true filamental solenoid. [7].

A formula for the mutual inductance between two solenoids modeled as current sheets, hinted in [7], is (adapting [5]):

$$M=\frac{2\pi n_1n_2}{h_1h_2}\left\{\begin{matrix}W(b_2-b_1+h_2)+W(b_2-b_1+h_1)-\\W(b_2-b_1+h_2-h_1)-W(b_2-b_1)\end{matrix}\right\};$$

$$W(x)=xW'(x)+\frac{8(r_1r_2)^{3/2}}{3k}\left[K-\left(\frac{2}{k^2}-1\right)(K-E)\right];$$

$$W'(x)=\frac{2x\sqrt{r_1r_2}}{k}(K-E)\pm$$

$$\pm\left|r_1^2-r_2^2\right|\left[KE(k',\theta)-(K-E)F(k',\theta)-\frac{\pi}{2}\right];$$

$$k=\sqrt{\frac{4r_1r_2}{x^2+(r_1+r_2)^2}};\ \ k'=\sqrt{1-k^2};$$

$$\theta=sin^{-1}\sqrt{\frac{1+\left(\frac{x}{r_1+r_2}\right)^2}{1+\left(\frac{x}{r_1-r_2}\right)^2}} \qquad (22)$$

The signal or the $\pm$ term is positive if x is positive. When $r_1=r_2$ and $x=0$ (coils touching), $k=1$, and the formula for $W(x)$ tends to a limit. Comparing (21) with (20), it can be seen that (22) can also be written using the complete elliptic integral of the third kind, that is easier to evaluate. Only the formula for $W'(x)$ changes:

$$W'(x)=x\left\{z(K-E)+\frac{(r_1-r_2)^2}{z}\left[K-\Pi(k,c)\right]\right\};$$

$$z=\sqrt{(r_1+r_2)^2+x^2}; \qquad (23)$$

$$k=\frac{2\sqrt{r_1r_2}}{z};\ \ c=\frac{2\sqrt{r_1r_2}}{r_1+r_2}$$

Another expression for $W'(x)$ is obtained recognizing that Heuman's Lambda function $\Lambda_0(k,\theta)$ appears in (22) (a restricted case is listed in [9]):

$$W'(x)=\frac{2x\sqrt{r_1r_2}}{k}(K-E)\pm\left|r_1^2-r_2^2\right|\frac{\pi}{2}\left[\Lambda_0(k,\theta)-1\right];$$

$$k=\sqrt{\frac{4r_1r_2}{x^2+(r_1+r_2)^2}};\ \ \theta=sin^{-1}\sqrt{\frac{1+\left(\frac{x}{r_1+r_2}\right)^2}{1+\left(\frac{x}{r_1-r_2}\right)^2}} \qquad (24)$$

The same equivalence can be used in (21). This just simplifies the notation. The derivation of (22) and other variations of it can be found in [14].

Other formulas for mutual inductances between cylindrical and flat coils, that sometimes are equivalent to to ones discussed above, can be found in ref. [9] (formulas involving current-sheet disk and solenoidal coils, in some particular arrangements), [10] (current-sheet disk-solenoid mutual inductance and a circular filament method), [11] (complicated formula for the mutual inductance between two rectangular coils) [12] (circular filament method for rectangular coils), and in the classical reference [13] (with many tables and references). Another interesting paper is [15], that contains alternative deductions, calculation methods, and equivalent forms for some of these equations.

Turns-independent coupling coefficient

When the coils are considered as current sheets, the coupling coefficient $k=M/\sqrt{L_1L_2}$ becomes independent from the numbers of turns in the coils. For solenoidal coils, for example, this happens if the inductances are calculated by Lorenz' formula (14) and the mutual inductance is calculated by Snow/Jones' formula (22).

V. Primary coils with all the turns in parallel

Low inductance primary coils can be built by connecting the turns of the coil in parallel instead of in series. Inductances and mutual inductances of a transformer built in this way can be calculated by the procedure:

1) Calculate the the inductance matrix of the whole system, considering each individual primary turn as a separate inductor. The program uses (3) for inductances and mutual inductances of the primary side, and for the inductance of the secondary coil. Mutual inductances between the primary turns and the secondary coils are obtained by (17). For n primary turns, this is an $(n+1)\times(n+1)$ matrix.

2) Invert the matrix, and add all the first n lines and columns. This corresponds to have the same voltage over all the primary turns, and a primary current that is the sum of the currents in all the turns.

3) Invert again the resulting 2×2 matrix, obtaining the equivalent primary and secondary inductances, and the mutual inductance.

A curious effect of this connection is that the secondary inductance is slightly reduced, because of the different mutual inductances between the primary turns and the secondary coil. The resulting mutual inductance is similar to the mutual inductance between two spiral coils, and the primary inductance is similar to the inductance of a single turn current sheet coil.

VI. Experimental results

Some solenoidal coils were wound with a copper tube and had their inductances measured. The table below compares the measured inductances with the prediction by Maxwell's method, with turns approximated by circular loops, and also lists the values that can be obtained with the formulas by Wheeler, Lorenz, Snow, and Kirchhoff. Inductances in μH, dimensions in meters.

Short coils with closely spaced turns: Coil radius = 0.486 m, tube diameter = 0.0095 m.

Height	N	Mea	Whe	Lor	Sno	Kir	Max
0.0921	5	49	44.03	49.58	49.20	49.23	49.36
0.0719	4	33	29.29	34.13	33.82	33.85	33.94
0.0516	3	20	17.16	21.02	20.75	20.78	20.84
0.0312	2	9	7.96	10.57	10.33	10.35	10.39
0.0109	1	2	2.08	3.28	3.00	3.03	3.03

Long coils with widely spaced turns: Coil radius = 0.486, tube diameter = 0.0095 m.

Height	N	Mea	Whe	Lor	Sno	Kir	Max
2.1336	5	17	9.07	9.09	18.75	18.17	18.17
1.7051	4	13	6.96	6.98	14.67	14.28	14.28
1.2764	3	10	4.90	4.90	10.64	10.42	10.42
0.8479	2	6	2.90	2.90	6.71	6.64	6.64
0.4191	1	3	1.09	1.09	3.00	3.03	3.03

The measurements show that the formulas based on a current sheet model (Lorenz' formula and its approximation by Wheeler), fail when the turns are widely spaced. The other formulas, based on filaments, however, work well in all cases.

A version of this document was published in [16].

Acknowledgment: Thanks to Godfrey Loudner for the hint about the AGM algorithm and several papers, and to Barton B. Anderson for the measurements.

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