

Capacitance Calculations

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Abstract—This document describes calculation methods for distributed capacitances of objects with several particular shapes, and methods for the evaluation of the electric fields and forces. It's fundamentally a collection of formulas, some not very easy to find in the literature. The algorithms were implemented in the Inca program, available at <http://www.coe.ufrj.br/~acmq/programs>.

I. INTRODUCTION

Most of the formulas below are known since long time, most dating from works in the XIX century. Some appear in Maxwell's book [1], and some in other collections of explicit formulas for electromagnetic problems, as [2], or in other early works as [3]-[5]. In most cases I have just adapted the notation, but some derivations not found in other works are presented too.

In most of the early works, capacitance is expressed in units of length. For example, the capacitance of a sphere of radius a in free space is listed in [1] and [2] as $C=a$. To convert this unit to Farads, it's necessary to multiply the value by $4\pi\epsilon_0$, where ϵ_0 is the permissivity of vacuum, $\epsilon_0 = 8.8541878 \times 10^{-12}$. ϵ_0 can be calculated from the speed of light c and from the magnetic permeability of vacuum, $\mu_0 = 4\pi \times 10^{-7}$ (a definition), from the relation:

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \quad (1)$$

The capacitance of a sphere of radius a meters is then:

$$C_{\text{sphere}} = 4\pi\epsilon_0 a = 111.26501a \text{ pF} \quad (2)$$

Other figures that have simple expressions for the free-space capacitance are:

A thin flat disk with radius a [2]:

$$C_{\text{disk}} = 8\epsilon_0 a = 70.833503a \text{ pF} \quad (3)$$

An open hemisphere with radius a [2]:

$$C_{\text{open hemisphere}} = 4\pi\epsilon_0 a(1/2+1/\pi) = 91.049254a \text{ pF} \quad (4)$$

A closed (with a flat disk) hemisphere with radius a [2]:

$$C_{\text{closed hemisphere}} = 8\pi\epsilon_0 a(1-1/\sqrt{3}) = 94.052249 \text{ pF} \quad (5)$$

Two spheres with radius a in contact [1]:

$$C_{\text{two spheres}} = 8\pi\epsilon_0 a \text{Ln}(2) = 154.24505a \text{ pF} \quad (6)$$

An "oblate spheroid" is the figure generated by the rotation of an ellipse around its minor axis. A "prolate spheroid" is generated by the rotation of an ellipse around its major axis. The capacitances of these figures are, considering the major axis with length $2a$ and the minor axis with the length $2b$ [2]:

$$C_{\text{oblate}} = 4\pi\epsilon_0 \frac{\sqrt{a^2-b^2}}{\sin^{-1} \frac{\sqrt{a^2-b^2}}{a}} \quad (7)$$

$$C_{\text{prolate}} = 4\pi\epsilon_0 \frac{\sqrt{a^2-b^2}}{\ln \frac{a+\sqrt{a^2-b^2}}{b}} \quad (8)$$

Note the limits when $a = b$ reducing to (2), and the reduction to (3) when $b = 0$ in (7).

For bodies embedded in materials with other permissivities, it's just a question of multiplying ϵ_0 by the relative permissivity ϵ of the material. The case when different dielectrics are present on the structure will be not discussed here.

II. CAPACITANCE OF A TOROID

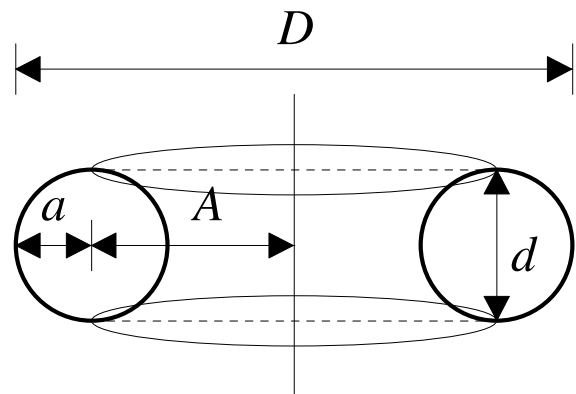


Fig. 1. Toroid with major diameter D , minor diameter d , center radius A , and tube radius a .

From [2] (the same formula appears in [3], that is probably the origin of this formula, but in a somewhat different notation) the capacitance of a toroid with major diameter D and minor diameter d , $d < D/2$, (fig. 1) is:

$$C = 16\epsilon_0 \sqrt{A^2 - a^2} \sum_{n=0}^{\infty} \sigma_n \frac{Q_{n-1/2}(x)}{P_{n-1/2}(x)}; \quad (9)$$

$$\sigma_n = 1/2 \text{ for } n = 0, 1 \text{ for } n > 0;$$

$$A = \frac{D-d}{2}; \quad a = \frac{d}{2}; \quad x = \frac{A}{a}$$

where $P_{n-1/2}(x)$ and $Q_{n-1/2}(x)$ are Legendre functions, or in this case, "toroidal functions".

These functions can be evaluated in the following way: The first two terms can be obtained from their relations with the complete elliptic integrals of first and second kinds:

$$Q_{-1/2}(x) = kK;$$

$$Q_{1/2}(x) = 2 \frac{K-E}{k} - kK;$$

$$P_{-1/2}(x) = \frac{2}{\pi} kK';$$

$$P_{1/2}(x) = \frac{2}{\pi} \left(\frac{2E'}{k} - kK' \right) \quad (10)$$

The modulus for the elliptic integrals K and E is:

$$k = \sqrt{\frac{2a}{A+a}} \quad (11)$$

And for the elliptic integrals K' and E' (evaluated in the same way, with modulus k'):

$$k' = \sqrt{1-k^2} \quad (12)$$

This is enough for the evaluation of the two first terms of the series (enough for thin toroids). The other terms can be obtained using the recursion for Legendre functions, identical for both functions:

$$P_{m+1/2}(x) = \frac{2mxP_{m-1/2}(x) - (m-1/2)P_{m-1/2-1}(x)}{m+1/2}; \quad (13)$$

$$Q_{m+1/2}(x) = \frac{2mxQ_{m-1/2}(x) - (m-1/2)Q_{m-1/2-1}(x)}{m+1/2}$$

where $m=n-1$. All the terms can then be easily computed, starting with $n=2$ in the series (9), or $m=1$.

The complete elliptic integrals are the irreducible functions:

$$K = F(k, \pi/2) = F(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} \quad (14)$$

$$E = E(k, \pi/2) = E(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \phi} d\phi$$

They can be quickly and precisely evaluated using the arithmetic-geometric mean method, below implemented in a Pascal routine:

```
{
Complete elliptic integrals of first
and second classes - AGM method.
Returns the global variables:
Ek=E(c) and Fk=F(c) (E and K)
```

```
Doesn't require more than 7 iterations for
c between 0 and 0.9999999999.
Reference: Pi and the AGM, J. Borwein and
P. Borwein, John Wiley & Sons.
```

```
}
procedure EF(c:real);
var
  a,b,al,bl,E,i:real;
begin
  a:=1;
  b:=sqrt(1-sqr(c));
  E:=1-sqr(c)/2;
  i:=1;
  repeat
    al:=(a+b)/2;
    bl:=sqrt(a*b);
    E:=E-i*sqr((a-b)/2);
    i:=2*i;
    a:=al;
    b:=bl;
  until abs(a-b)<1e-15;
  Fk:=pi/(2*a);
  Ek:=E*Fk
end;
```

III. APPROXIMATE CALCULATIONS FOR PARTIAL TOROIDS

A partial toroid can be described as a surface generated by the revolution of a partial circle of radius a centered at a distance A along the radial axis r from the revolution axis z . The circle limits are defined by two angles θ_1 and θ_2 . See fig. 2.

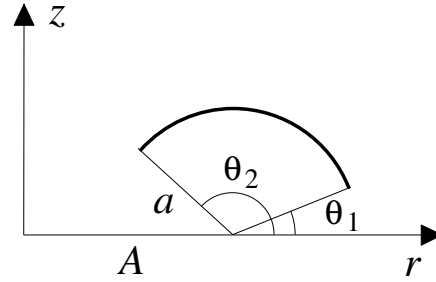


Fig. 2: A partial toroidal surface is generated by the rotation of a partial circle around the vertical axis.

With this formulation several figures can be generated, as a regular toroid when $\theta_2 - \theta_1 = 2\pi$ and $a < A$, a sphere when $A = 0$, $\theta_2 = -\theta_1 = \pi/2$, an open hemisphere, etc. Even overlapping toroids, with $a > A$, can be generated.

This surface can be decomposed in a set of n infinitely thin circles with axes at the z axis, positioned at heights z_i , and with radii r_i , uniformly spaced at angles $\Delta\theta$ along the surface:

$$\Delta\theta = \frac{\theta_2 - \theta_1}{n}$$

$$\theta_i = \theta_1 + \frac{\Delta\theta}{2} + (i-1)\Delta\theta, \quad i = 1 \dots n \quad (15)$$

$$r_i = A + a \cos \theta_i$$

$$z_i = a \sin \theta_i$$

Each of these rings has a uniform charge distribution, with a total charge q_i . The potential Ψ due to each ring i at any given position r_0, z_0 is given by:

$$\Psi_i(r_0, z_0) = \frac{q_i}{4\pi^2 \epsilon_0 \sqrt{|r_0 r_i|}} Q_{-1/2} \left(\frac{2}{k^2} - 1 \right) = \frac{q_i}{2\pi^2 \epsilon_0 R_1} K$$

$$k = \frac{2\sqrt{|r_0 r_i|}}{R_1} \quad (16)$$

$$R_1 = \sqrt{(|r_i| + |r_0|)^2 + (z_0 - z_i)^2}$$

The absolute values allow correct treatment of the cases when some radii are negative.

Considering then the mutual influences among all the rings, a matrix \mathbf{P} can be computed, that allows the calculation of the potentials v_i at each ring, once the charges q_i are known [1]:

$$\mathbf{v} = \mathbf{P}\mathbf{q}$$

$$P_{ij} = P_{ji} = \Psi_i(r_j, z_j) / q_j \quad (17)$$

$$P_{ii} = \Psi_i(r_i, z_i + R) / q_i$$

For the calculation of the “self-potentials” P_{ii} , something must be assumed about the radius of the rings, R . The formulation calculates then the potentials at a distance R above the rings. The maximum physically possible value of R would be when adjacent rings touch:

$$R_{\max} = a \sin \frac{\Delta\theta}{2} \quad (18)$$

Any reasonable fraction of this value can be used with similar results, but there is one that produces better results in the next calculation, that was found (by trying!) to, curiously, be:

$$R = \frac{a}{\pi} \sin \frac{\Delta\theta}{2} \quad (19)$$

This radius makes the area of the surface of the ring to be identical to the flat area represented by it, at least in the cases when $\theta = n\pi/2$ and small $\Delta\theta$ (as in the equator and poles of a sphere split in many rings).

The charge distribution for uniform potential V at all the rings can be calculated by inverting the matrix \mathbf{P} . The total charge in each ring is then obtained from a sum of the corresponding lines of the inverse of \mathbf{P} , \mathbf{C} . The coefficients of \mathbf{C} are the influence coefficients k_{ij} :

$$q_i = V \sum_{j=1}^n k_{ij} \quad (20)$$

$$\mathbf{C} = \mathbf{P}^{-1}$$

And the capacitance of the whole assembly is simply the sum of all the elements of \mathbf{C} :

$$C_{\text{total}} = \sum_{i=1}^n \sum_{j=1}^n k_{ij} \quad (21)$$

Surface electric field

The electric field at any point of the surface is normal to it and can be calculated by Gauss' law as proportional to the charge density at that point of the surface:

$$E_i = \frac{\rho_i}{\epsilon_0} \quad (22)$$

where ρ_i is the surface charge density, uniform around the ring i . For a closed surface, the electric field is entirely at the outer surface. In this case, it can be calculated directly from the charge distribution alone.

Assuming constant voltage at the surface of the object, the charges at the rings can be calculated by (20). The ring i has a length $2\pi r_i$ and a total charge q_i . The ring represents a thin belt with width equal to $a\Delta\theta$. The charge density and the electric field in a small length l are then:

$$\rho_i = \frac{q_i}{2\pi r_i} \frac{1}{la\Delta\theta} = \frac{q_i}{2\pi r_i a \Delta\theta}$$

$$E_i = \frac{q_i}{2\pi r_i a \Delta\theta \epsilon_0} \quad (23)$$

An important application of this calculation is the determination of the breakout voltage of the object, the voltage that causes ionization of the air around it when the electric field reaches about 3 MV/m:

$$V_{\max} = 3000 / \text{Max } E_i \text{ kV} \quad (24)$$

For a toroid, this value occurs at the maximum diameter. In the case of open objects, it's not possible to calculate the surface electric field in this way, because it is split in an unknown way between the two sides of the surface. The calculation is also meaningless if the object has an edge, where the electric field is ideally infinite. A strange problem with (23) is that it fails when the rings are close to the center of a spherical surface. The last ring appears to have significantly less charge than it should have (around 92%). The calculations for capacitance, however, continue to result in good values.

IV. GENERAL TRUNCATED CONES

Any other figure with circular symmetry can be analyzed by the same method. A simple case is the revolution of a straight line around the central axle, that generates figures ranging from a flat disk with a possible central hole to a cone or an open cylinder.

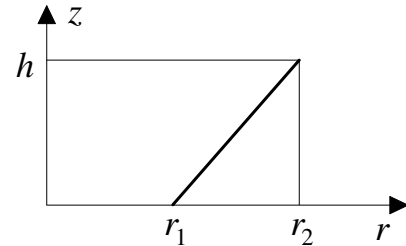


Fig. 3. A line that rotates around the vertical axis.

The coordinates of the rings are then:

$$\begin{aligned}
\Delta z &= \frac{h}{n} \\
\Delta r &= \frac{r_2 - r_1}{n} \\
z_i &= \frac{\Delta z}{2} + (i-1)\Delta z, \quad i = 1 \dots n \\
r_i &= r_1 + \frac{\Delta r}{2} + (n-1)\Delta r, \quad i = 1 \dots n
\end{aligned} \tag{25}$$

The radius to use in the calculation of the self-potentials would be, still using the maximum divided by π :

$$R = \frac{\sqrt{(r_2 - r_1)^2 + h^2}}{2n\pi} \tag{26}$$

With this radius, the surface charge density and the surface electric field (for a closed object) can be calculated considering that the surface area of the ring is identical to the belt area represented by it, what is approximately valid also for the case of curves, using R given by (19):

$$\begin{aligned}
\rho_i &= \frac{q_i - l}{2\pi r_i} \frac{1}{l2\pi R} = \frac{q_i}{4\pi^2 r_i R} \\
E_i &= \frac{q_i}{4\pi^2 r_i R \epsilon_0}
\end{aligned} \tag{27}$$

V. ELECTRIC FIELD FROM A RING

The electric field anywhere can be calculated by adding the electric fields due to the rings. From (16), the radial and axial components of the electric field can be calculated by differentiation, resulting in:

$$\begin{aligned}
E_{radial} &= -\frac{d\Psi_i}{dr_0} = \\
&= -\frac{q_i \text{sign } r_i}{2\pi^2 \epsilon_0 R_1^3} \left(-(|r_i| - |r_0|)K + \frac{E - k'^2 K}{kk'^2} \left(\frac{2|r_0|}{k} - k(|r_i| - |r_0|) \right) \right)
\end{aligned} \tag{28}$$

$$E_{axial} = -\frac{d\Psi_i}{dz_0} = \frac{q_i(z_i - z_0)E}{2\pi^2 \epsilon_0 R_1^3 k'^2} \tag{29}$$

$$E_{total} = \sqrt{E_{radial}^2 + E_{axial}^2} \tag{30}$$

where the derivative of the elliptic integral K in relation to the modulus k was used (the derivative of E is listed below too for reference, but was not necessary):

$$\begin{aligned}
\frac{dK}{dk} &= \frac{E - k'^2 K}{kk'^2}; \quad \frac{dE}{dk} = \frac{E - K}{k} \\
k'^2 &= 1 - k^2
\end{aligned} \tag{31}$$

VI. GENERAL CASE WITH AXIAL SYMMETRY

The capacitance matrix and the potential and electric field around a series of objects with axial symmetry decomposed in thin rings can then be easily calculated. The objects are decomposed in series of partial toroids conical sheets, and other shapes (as ellipses) and these parts are decomposed in rings. To obtain the

capacitance matrix, it's just a question of adding the terms of the total capacitance matrix that correspond to the rings that belong to the objects, instead of adding them all to obtain the capacitance of the entire object. The charges in all the rings can be obtained from the complete equation $\mathbf{q} = \mathbf{C}\mathbf{V}$, with the assigned voltages in the objects arranged in \mathbf{V} in correspondence with the rings that belong to the objects. The potential anywhere around the objects is obtained by adding (16) for all the rings, and the electric field by adding (28) and (29) and using (30). The terms at the diagonal of the capacitance matrix correspond to the capacitances of the objects to ground when all the other objects are grounded too. The influence coefficients out of the diagonal measure the relation between the charge induced in one object and the voltage in another, when all the other objects are grounded. From the capacitance matrix, a model of the circuit using lumped capacitors can be derived, by observing the equivalence:

$$\begin{aligned}
\mathbf{C} &= \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{12} & k_{22} & \dots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1n} & k_{2n} & \dots & k_{nn} \end{bmatrix} = \\
&= \begin{bmatrix} C_1 + C_{12} + \dots + C_{1n} & -C_{12} & \dots & -C_{1n} \\ -C_{12} & C_2 + C_{12} + \dots + C_{2n} & \dots & -C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -C_{1n} & -C_{2n} & \dots & C_n + C_{n-1} + \dots + C_{1n} \end{bmatrix}
\end{aligned} \tag{32}$$

C_1, C_2, \dots, C_n are direct capacitances between the elements and the ground, and the other elements are the negative of the floating capacitances between the objects. The direct capacitance to ground for the object i is just the sum of the elements in the line, or column, i of \mathbf{C} .

VII. MAXIMUM ELECTRIC FIELD BETWEEN TWO SPHERES

A good test for these field calculations is the known formula for the maximum electric field between two different spheres [4]. The expression comes directly from the method of images developed by Lord Kelvin [10]. For two spheres of radii a and b , $a < b$, with distance between centers c , at potentials v_1 and v_2 , the maximum electric field at the surface of the smaller sphere (assumed as being where the surface of the smaller sphere intercepts the line between the centers of the spheres) is given by:

$$\begin{aligned}
E_{max} &= \\
&= \frac{(1+\xi)^2}{a(1-\xi)} \left\{ v_1 \left[\frac{1-\xi}{(1+\xi)^2} + \alpha \frac{1-\xi\alpha^2}{(1+\xi\alpha^2)^2} + \alpha^2 \frac{1-\xi\alpha^4}{(1+\xi\alpha^4)^2} + \dots \right] \right. \\
&\quad \left. - v_2 \left[\eta \frac{1-\eta\alpha}{(1+\eta\alpha)^2} + \alpha\eta \frac{1-\eta\alpha^3}{(1+\eta\alpha^3)^2} + \alpha^2\eta \frac{1-\eta\alpha^5}{(1+\eta\alpha^5)^2} + \dots \right] \right\} \\
\xi &= \frac{a+b\alpha}{c}; \quad \eta = \frac{b+a\alpha}{c}; \\
\alpha &= \text{root} < 1 \text{ of } (a\alpha+b)(b\alpha+a) = c^2\alpha
\end{aligned} \tag{33}$$

This formula converges slowly when the spheres are at small distance, but the speed is acceptable. [4]

develops a better expression for the case of spheres at small distance too. The choice of α appears to work correctly too when the other root is used. The formula works for any choice of v_1 and v_2 , but always calculates the electric field at the point of the sphere with radius a closest to the other sphere, even when this is not the point of maximum electric field (as when v_1 and v_2 have the same sign). Note that at that point the calculation using rings, as formulated, calculates an electric field slightly smaller than the correct value.

VIII. CAPACITANCES OF TWO SPHERES

Similar formulas, due to Kirchhoff, lead to the capacitance matrix of two spheres [5]. For two spheres with radii a and b and distance between centers c :

$$\begin{aligned} k_{11} &= 8\pi\epsilon_0\lambda \left[\frac{\xi}{1+\xi^2} + \frac{\alpha\xi}{1+\alpha^2\xi^2} + \frac{\alpha^2\xi}{1+\alpha^4\xi^2} + \dots \right]; \\ k_{22} &= 8\pi\epsilon_0\lambda \left[\frac{\eta}{1+\eta^2} + \frac{\alpha\eta}{1+\alpha^2\eta^2} + \frac{\alpha^2\eta}{1+\alpha^4\eta^2} + \dots \right]; \\ k_{12} &= -8\pi\epsilon_0\lambda \left[\frac{\alpha}{1+\alpha^2} + \frac{\alpha^2}{1+\alpha^4} + \frac{\alpha^3}{1+\alpha^6} + \dots \right]; \\ \xi &= \sqrt{1 + \frac{\lambda^2}{a^2}} - \frac{\lambda}{a}; \quad \alpha = \frac{c\xi - a}{b}; \quad \eta = \frac{\alpha}{\xi}; \\ \lambda &= \frac{\sqrt{(c+a+b)(c-a-b)(c+a-b)(c-a+b)}}{2c} \end{aligned} \quad (34)$$

These formulas also converge slowly when the spheres are at small distance. [5] shows better formulas for small distances. [11][12] have the exact solution when the spheres are touching, eq. (49).

The coefficients of the capacitance matrix represent the ratio between the induced charges and the voltages. k_{11} and k_{12} represent capacitances to ground from a sphere with the other sphere grounded, and $-k_{12}$ is the floating capacitance between the spheres. The differential capacitance between the spheres is obtained by assuming opposite charges $\pm q$ on them:

$$C_{diff} = \frac{v_1 - v_2}{q} = \frac{k_{11}k_{22} - k_{12}^2}{k_{11} + k_{22} + 2k_{12}} \quad (35)$$

The capacitances to ground with the other sphere floating can also be calculated, by assuming zero charge in the floating sphere:

$$C_{10} = \frac{k_{11}k_{22} - k_{12}^2}{k_{22}}; \quad C_{20} = \frac{k_{11}k_{22} - k_{12}^2}{k_{11}} \quad (36)$$

The capacitance of both spheres to ground is simply the sum of the four coefficients, because both are at the same potential and the total charge appears in the ratio $C = Q/V$. This capacitance can be used to find the relation between total charge and voltage in a pith-ball electroscope. For equal spheres it varies between (6) when the spheres are touching and two times (2) when they are far apart:

$$C_{pair} = k_{11} + k_{22} + 2k_{12} \quad (37)$$

IX. POTENTIAL AND ELECTRIC FIELD AROUND A TOROID

The solution of this problem can be traced to [3]. The formula for the potential also appears in [2]. The potential around an isolated toroid in free space, with central radius A and tube radius a , at a radial distance r and axial distance z from the center, is found as:

$$\begin{aligned} \Psi(\alpha, \beta) &= \frac{2V}{\pi} \sqrt{2(\cosh \beta - \cos \alpha)} \sum_{n=0}^{\infty} \sigma_n \frac{Q_{n-1/2}(x)}{P_{n-1/2}(x)} P_{n-1/2}(\cosh \beta) \cos n\alpha; \\ \sigma_n &= 1/2 \text{ for } n=0, 1 \text{ for } n>0; \\ x &= \frac{A}{a}; \quad c = \sqrt{A^2 - a^2}; \\ \beta &= \frac{1}{2} \ln \frac{z^2 + (r+c)^2}{z^2 + (r-c)^2}; \\ \sin \alpha &= \frac{z}{r} \sinh \beta; \quad \cos \alpha = \cosh \beta - \frac{c}{r} \sinh \beta \\ \alpha &= \tan^{-1} \frac{\sin \alpha}{\cos \alpha} = \tan^{-1} \frac{2cz}{r^2 + z^2 - c^2}; \end{aligned} \quad (38)$$

The surface electric field can be found by the differentiation of (38). The maximum occurs when $\cosh \beta = x = A/a$ (toroid surface) and $\alpha=0$ (major diameter). The result, hinted in [3] but not developed, is the series:

$$E = \frac{4\sqrt{2}V(x - \cos \alpha)^{3/2}}{\pi d(x^2 - 1)} \sum_{n=0}^{\infty} \sigma_n \frac{\cos n\alpha}{P_{n-1/2}(x)} \quad (39)$$

The ideal exact breakdown voltage can then be obtained as in (24). This series converges somewhat more slowly than (9) but still can achieve high precision. The series (38) may lose precision due to errors in the evaluation of $Q_{n+1/2}(x)$ by the recursion (13).

X. OTHER CAPACITANCES

Other capacitances, of shapes without circular symmetry, are known from numerical analysis. Classical cases are the capacitances of the square and triangular flat plates [6], the cube [7] and the tetrahedron [6], all with side a :

$$C_{\text{square}} = 4\pi\epsilon_0 a \times 0.348 = 38.7 \text{ a pF} \quad (40)$$

$$C_{\text{triangle}} = 4\pi\epsilon_0 a \times 0.25096 = 27.293 \text{ a pF} \quad (41)$$

$$C_{\text{cube}} = 4\pi\epsilon_0 a \times 0.6606782 = 73.51036 \text{ a pF} \quad (42)$$

$$C_{\text{tetrahedron}} = 4\pi\epsilon_0 a \times 0.3569 = 39.71 \text{ a pF} \quad (43)$$

XI. FORCES BETWEEN RINGS

The force between thin coaxial filaments can be calculated by an integration of the Coulomb force between them. For two rings with radii A and a and separation b , containing charges q_1 and q_2 , the force is given by the double integral (44). The term $\cos \theta$ projects the force along the axis:

$$\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0} \iint \frac{\cos\theta}{r^2} dq dq'; \\
r &= \sqrt{A^2 + a^2 + b^2 - 2Aa \cos(\varphi - \varphi')}; \\
\cos\theta &= b/r; \\
dq &= \frac{q_1}{2\pi} d\varphi; \\
dq' &= \frac{q_2}{2\pi} d\varphi'; \\
F_{12} &= \frac{q_1 q_2}{16\pi^3 \epsilon_0} \int_0^{2\pi} \int_0^{2\pi} \frac{bd\varphi d\varphi'}{\left(\sqrt{A^2 + a^2 + b^2 - 2Aa \cos(\varphi - \varphi')}\right)^3}
\end{aligned} \quad (44)$$

This integral can be exactly solved in terms of the complete elliptic integral of the third kind (45), which is also easy to evaluate. The code below computes in few iterations the three complete elliptic integrals $E(k)$, $F(k)$, and $\Pi(k,c)$:

$$\Pi(k,c) = \int_0^{\pi/2} \frac{d\varphi}{(1-c^2 \sin^2 \varphi) \sqrt{1-k^2 \sin^2 \varphi}} \quad (45)$$

```

{
Complete elliptic integrals of first, second,
and third kinds - AGM
Returns the global variables Ek=E(k), Fk=F(k),
and IIkc=II(k,c)
Reference: Garrett, Journal of Applied
Physics, 34, 9, 1963, p. 2571
}
procedure EFII(k,c:real);
var
a,b,d,e,f,a1,b1,d1,e1,f1,S,i:real;
begin
a:=1;
b:=sqrt(1-sqr(k));
d:=(1-sqr(c))/b;
e:=sqr(c)/(1-sqr(c));
f:=0;
i:=1/2;
S:=i*sqr(a-b);
repeat
a1:=(a+b)/2;
b1:=sqrt(a*b);
i:=2*i;
S:=S+i*sqr(a1-b1);
d1:=b1/(4*a1)*(2+d+1/d);
e1:=(d*e+f)/(1+d);
f1:=(e+f)/2;
a:=a1;
b:=b1;
d:=d1;
e:=e1;
f:=f1;
until (abs(a-b)<1e-15) and (abs(d-1)<1e-15);
Fk:=pi/(2*a);
Ek:=Fk-Fk*(sqr(k)+S)/2;
IIkc:=Fk*f+Fk;
end;

```

The solution for the force is:

$$\begin{aligned}
F_{12} &= \frac{q_1 q_2 b}{2\pi^2 \epsilon_0 r_1^3} \Pi(k,c) \\
k &= c = \frac{2\sqrt{Aa}}{r_1} \\
r_1 &= \sqrt{(A+a)^2 + b^2}
\end{aligned} \quad (46)$$

The formula reduces to the well known case of a charged ring and a point charge if one ring has zero radius. In this case $k=c=0$ and $\Pi(0,0)=\pi/2$:

$$F_{12} = \frac{q_1 q_2 b}{2\pi^2 \epsilon_0 (A^2 + b^2)^{3/2}} \frac{\pi}{2} = \frac{q_1 q_2 b}{4\pi \epsilon_0 (A^2 + b^2)^{3/2}} \quad (47)$$

To calculate the forces in conductors decomposed in rings, the charges in the rings are first calculated using $\mathbf{q}=\mathbf{C}\mathbf{V}$ and then the forces in each ring can be obtained by (46), by adding the values obtained between a given ring and all the others. Finally, the forces in all the rings belonging to each conductor are added. The calculations for rings belonging to a single conductor can be omitted, because they add to zero (and this is a good test of the algorithm).

The same result can be obtained by computing the forces multiplying the total axial electric field seen by a ring by its charge, using (29):

$$\begin{aligned}
F_{12} &= \frac{q_1 q_2 b}{2\pi^2 \epsilon_0 r_1^3 (1-k^2)} E(k) \\
k &= \frac{2\sqrt{Aa}}{r_1} \\
r_1 &= \sqrt{(A+a)^2 + b^2}
\end{aligned} \quad (48)$$

XII. EXAMPLES

Some toroids analyzed by the methods above. V_{\max} was obtained from (39) and (24), except for the ‘‘holeless’’ toroid, where (23) and (24) were used. All the capacitances (in this and the other examples) in pF:

| $D \times d$ | C_{exact} | 20 rings | 200 rings | V_{\max} (kV) |
|--------------|--------------------|-----------|-----------|-----------------|
| 0.2x0.1 | 9.6877342 | 9.6862459 | 9.6877328 | 226.2 |
| 0.3x0.1 | 13.527991 | 13.526517 | 13.527990 | 282.9485 |
| 0.4x0.1 | 17.200315 | 17.198812 | 17.200313 | 328.9148 |
| 0.5x0.1 | 20.738038 | 20.736480 | 20.738037 | 367.4999 |

Open hemispheres (D = diameter):

| D | C_{exact} | 20 rings | 200 rings |
|-----|--------------------|-----------|-----------|
| 0.2 | 9.1049254 | 9.0451871 | 9.0989244 |
| 0.3 | 13.657388 | 13.567781 | 13.648387 |
| 0.4 | 18.209851 | 18.090374 | 18.197849 |
| 0.5 | 22.762314 | 22.612968 | 22.747311 |

Flat disks (D = diameter):

| D | C_{exact} | 20 rings | 200 rings |
|-----|--------------------|-----------|-----------|
| 0.2 | 7.0833502 | 7.0067052 | 7.0757027 |
| 0.3 | 10.625025 | 10.510058 | 10.613554 |
| 0.4 | 14.166701 | 14.013411 | 14.151405 |
| 0.5 | 17.708376 | 17.516763 | 17.689257 |

Hollow cylinders (D = diameter, h = height):

| D | h | 20 rings | 200 rings |
|-----|-----|------------|------------|
| 0.2 | 1 | 27.2508153 | 27.5562772 |
| 0.3 | 1 | 32.7125753 | 33.0502066 |
| 0.4 | 1 | 37.6883716 | 38.0515957 |
| 0.5 | 1 | 42.3659124 | 42.7508210 |

Hollow cones (D = diameter, h = height):

| D | h | 20 rings | 200 rings |
|-----|---|------------|------------|
| 0.2 | 1 | 20.6332474 | 20.8219907 |
| 0.3 | 1 | 24.3200548 | 24.5554254 |
| 0.4 | 1 | 27.7305014 | 28.0027331 |
| 0.5 | 1 | 30.9951485 | 31.3004301 |

In the last two cases no explicit formulas were found in the literature, although very probably they are known.

The general algorithm for objects with axial symmetry was implemented in the Inca program and used to generate the next examples:

A closed hemisphere can be generated by the combination of an open hemisphere and a flat disk (half of the rings for each element, D = diameter):

| D | C_{exact} | 20 rings | 200 rings |
|-----|--------------------|------------|------------|
| 0.2 | 9.4052249 | 9.3751321 | 9.4038325 |
| 0.3 | 14.1078374 | 14.0626982 | 14.1057488 |
| 0.4 | 18.8104499 | 18.7502642 | 18.8076651 |
| 0.5 | 23.5130623 | 23.4378303 | 23.5095813 |

Two spheres in contact (D = diameter, half of the rings for each sphere):

| D | C_{exact} | 20 rings | 200 rings |
|-----|--------------------|------------|------------|
| 0.1 | 7.7123025 | 7.7105894 | 7.7123007 |
| 0.2 | 15.4246050 | 15.4211788 | 15.4246014 |
| 0.3 | 23.1369075 | 23.1317682 | 23.1369021 |
| 0.4 | 30.8492100 | 30.8423576 | 30.8492028 |
| 0.5 | 38.5615125 | 38.5529470 | 38.5615035 |

Two different spheres in contact. Some cases are listed in [11][12]. For spheres of radii a and b , when $a = 2b$ $C = 4\pi\epsilon_0 a \text{Ln} 3$, and when $a = 3b$ $C = 4\pi\epsilon_0 a 3/4 \text{Ln} 4$.

| a, b | Exact | 40 rings | 400 rings |
|----------------|------------|------------|------------|
| 0.1, 0.05 | 12.2237103 | 12.2233396 | 12.2237099 |
| 0.1, 0.0333... | 11.5684538 | 11.5680790 | 11.5684534 |
| 0.1, 0.025 | 11.3481786 | 11.3477839 | 11.3481782 |

The last case, $a = 4b$, is more complicated, $C = 4\pi\epsilon_0 a (1/2 \text{Ln} 5 + 2\sqrt{5}/10 \text{Ln}((1+\sqrt{5})/2))$. The general case can be computed by the formula in [11] (in a different form) and [12]:

$$C = 4\pi\epsilon_0 \frac{ab}{a+b} \left(-\psi\left(\frac{b}{a+b}\right) - \psi\left(\frac{a}{a+b}\right) - 2\gamma \right) \quad (49)$$

where $\psi(x)$ is the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$, derivative of the logarithm of the gamma function, and γ is Euler's constant, $\gamma \cong 0.577215665$.

A toroid with the central hole closed by a thin disk. Note the small difference to a regular toroid. A toroid where the closure of the central hole doubles the capacitance would have an aspect ratio of about 1×0.0004 . Half of the rings for each element, D = major diameter, d = diameter of the tube:

| D x d | 20 rings | 200 rings | 400 rings |
|---------|------------|------------|------------|
| 0.3x0.1 | 13.5176679 | 13.5296046 | 13.5296149 |
| 0.4x0.1 | 17.2225678 | 17.2348074 | 17.2348180 |
| 0.5x0.1 | 20.8623320 | 20.8748605 | 20.8748714 |

Maximum electric field between spheres with opposite voltages. Half of the rings to each sphere. Dimensions as in (33). Fields in V/m/V:

| a, b, c | Exact | 40 rings | 400 rings |
|---------------|------------|-----------|-----------|
| 0.1, 0.1, 0.5 | 14.7654541 | 14.654655 | 14.762658 |
| 0.1, 0.2, 0.5 | 20.7165237 | 20.434842 | 20.711307 |
| 0.1, 0.3, 0.5 | 32.2318226 | 31.394734 | 32.219293 |

Capacitance matrix for two spheres. Half of the rings to each sphere. Dimensions as in (33):

k_{11} (radius a)

| a, b, c | Exact | 40 rings | 400 rings |
|---------------|------------|------------|------------|
| 0.1, 0.1, 0.5 | 11.6112177 | 11.6108704 | 11.6112174 |
| 0.1, 0.2, 0.5 | 12.3051750 | 12.3047650 | 12.3051745 |
| 0.1, 0.3, 0.5 | 13.7605384 | 13.7603742 | 13.7605373 |

k_{22} (radius b)

| a, b, c | Exact (pF) | 40 rings | 400 rings |
|---------------|------------|------------|------------|
| 0.1, 0.1, 0.5 | 11.6112177 | 11.6108704 | 11.6112174 |
| 0.1, 0.2, 0.5 | 24.3154312 | 24.3146700 | 24.3154303 |
| 0.1, 0.3, 0.5 | 38.6334041 | 38.6326963 | 38.6334025 |

k_{12}

| a, b, c | Exact (pF) | 40 rings | 400 rings |
|---------------|------------|------------|------------|
| 0.1, 0.1, 0.5 | -2.3264588 | -2.3263316 | -2.3264587 |
| 0.1, 0.2, 0.5 | -4.9456676 | -4.9454137 | -4.9456673 |
| 0.1, 0.3, 0.5 | -8.3626059 | -8.3626805 | -8.3626051 |

Force between the two halves of a charged sphere. The exact value is simply $F = \pi\epsilon_0 v^2 / 2$, independent of the radius of the sphere. The approximate calculation is also independent of it. Values for 1 V at the sphere.

| Radius (m) | Exact (pN) | 40 rings | 400 rings |
|------------|------------|----------|-----------|
| 1 | 13.90813 | 13.56727 | 13.87342 |

Force between two equal spheres at the same potential, of 1 V. The exact values were computed by the approximations in [8] and [9]. The exact value for spheres in contact is $F = 4\pi\epsilon_0 v^2 (\text{Ln} 2 - 1/4) / 6$ [10].

| a, b, c | Exact (pN) | 40 rings | 400 rings |
|---------------|------------|----------|-----------|
| 0.1, 0.1, 0.2 | 8.217796 | 8.218122 | 8.217796 |
| 0.1, 0.1, 0.5 | -2.996884 | 2.996774 | 2.996872 |
| 0.1, 0.1, 1.0 | -0.915993 | 0.915949 | 0.915993 |

Force between the two halves of a horned toroid at 1 V. This case apparently has no known exact solution. The force doesn't depend on the size of the device and is proportional to the square of the potential, as happens in all cases of objects in contact. Forces in pN

| D (m) | 40 rings | 200 rings | 400 rings |
|-------|----------|-----------|-----------|
| 1 | 15.55986 | 16.02683 | 16.08724 |

Force between two stacked horned toroids in contact, at 1V. The exact solution is unknown, but in this case the

numerical analysis is expected to be precise. Forces in pN.

| $D(m)$ | 40 rings | 200 rings | 400 rings |
|--------|----------|-----------|-----------|
| 1 | 10.53158 | 10.53005 | 10.53003 |

The force increases as the aspect ratio of the toroids increase. It doubles for a 12.6874×1 toroid.

The problem with this approach is that as the number of rings increases it becomes more and more difficult to invert the matrix \mathbf{P} with precision and in reasonable time. The examples show that the method is not very precise for objects with edges. The precision can be enhanced by adding more rings to the regions close to edges. For example, in the example of force between the two halves of a sphere, if the ± 10 degrees around the equator of the sphere are modeled with 300 rings, with 100 rings for the remaining surface, the obtained force is of 13.90274 pN, with 4 correct digits. For the horned toroid, the same distribution produces 16.14268 pN, or $4\pi\epsilon_0 \times 0.1450832$ N, possibly with similar precision.

In the last page is a table of exact toroid capacitances calculated by (9). Note that it would be enough to have a single column with normalized aspect ratios, since for a fixed aspect ratio the capacitance is directly proportional to the major (or minor) diameter.

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This document is not a published paper.

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CHANGES

9/2/2011: Corrected eq. 9 and small corrections in the text.

6/1/2012: Corrected eq. 33.

9/1/2012: Added section IX.

10/2/2012: Added eq. 37.

15/2/2012: Added section about forces.

24/2/2012: Added formula for the capacitance of two different spheres in contact.

25/2/2012: Added more examples of forces, small corrections.

Exact toroid capacitances (diameters in meters, capacitances in pF)

| Minor d. Major d. | 0.100 | 0.110 | 0.120 | 0.130 | 0.140 | 0.150 | 0.160 | 0.170 | 0.180 |
|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.100 | 3.707 | 4.148 | 4.431 | 4.653 | 4.833 | 5.000 | 5.156 | 5.303 | 5.441 |
| 0.125 | 4.468 | 5.001 | 5.340 | 5.598 | 5.816 | 6.010 | 6.181 | 6.338 | 6.481 |
| 0.150 | 5.205 | 6.221 | 6.518 | 6.764 | 6.979 | 7.174 | 7.349 | 7.506 | 7.645 |
| 0.175 | 5.923 | 6.630 | 7.080 | 7.417 | 7.691 | 7.929 | 8.143 | 8.339 | 8.511 |
| 0.200 | 6.625 | 7.414 | 7.919 | 8.295 | 8.604 | 8.861 | 9.094 | 9.306 | 9.492 |
| 0.225 | 7.313 | 8.182 | 8.740 | 9.156 | 9.492 | 9.778 | 10.029 | 10.258 | 10.469 |
| 0.250 | 7.990 | 8.937 | 9.547 | 10.000 | 10.369 | 10.679 | 10.951 | 11.196 | 11.422 |
| 0.275 | 8.657 | 9.679 | 10.340 | 10.834 | 11.232 | 11.567 | 11.860 | 12.122 | 12.362 |
| 0.300 | 9.316 | 10.411 | 11.121 | 11.654 | 12.082 | 12.443 | 12.757 | 13.037 | 13.292 |
| 0.325 | 9.966 | 11.133 | 11.892 | 12.462 | 12.921 | 13.307 | 13.642 | 13.940 | 14.211 |
| 0.350 | 10.609 | 11.846 | 12.653 | 13.260 | 13.749 | 14.160 | 14.517 | 14.838 | 15.119 |
| 0.375 | 11.246 | 12.551 | 13.405 | 14.048 | 14.567 | 15.003 | 15.381 | 15.716 | 16.019 |
| 0.400 | 11.876 | 13.250 | 14.149 | 14.828 | 15.376 | 15.838 | 16.237 | 16.590 | 16.909 |
| 0.425 | 12.502 | 13.941 | 14.886 | 15.600 | 16.177 | 16.663 | 17.084 | 17.456 | 17.791 |
| 0.450 | 13.122 | 14.627 | 15.616 | 16.365 | 16.970 | 17.481 | 17.922 | 18.313 | 18.664 |
| 0.475 | 13.737 | 15.306 | 16.340 | 17.122 | 17.756 | 18.291 | 18.753 | 19.162 | 19.530 |
| 0.500 | 14.348 | 15.981 | 17.057 | 17.874 | 18.535 | 19.093 | 19.577 | 20.005 | 20.389 |
| 0.525 | 14.955 | 16.650 | 17.769 | 18.619 | 19.308 | 19.890 | 20.394 | 20.840 | 21.240 |
| 0.550 | 15.558 | 17.315 | 18.476 | 19.358 | 20.075 | 20.680 | 21.204 | 21.668 | 22.085 |
| 0.575 | 16.157 | 17.975 | 19.177 | 20.092 | 20.836 | 21.464 | 22.009 | 22.492 | 22.924 |
| 0.600 | 16.753 | 18.631 | 19.874 | 20.821 | 21.591 | 22.242 | 22.807 | 23.307 | 23.756 |
| 0.625 | 17.346 | 19.284 | 20.567 | 21.546 | 22.342 | 23.016 | 23.601 | 24.118 | 24.583 |
| 0.650 | 17.935 | 19.932 | 21.256 | 22.265 | 23.088 | 23.784 | 24.388 | 24.924 | 25.406 |
| 0.675 | 18.522 | 20.577 | 21.940 | 22.981 | 23.829 | 24.547 | 25.171 | 25.724 | 26.212 |
| 0.700 | 19.105 | 21.211 | 22.621 | 23.692 | 24.566 | 25.306 | 25.949 | 26.520 | 27.032 |
| 0.725 | 19.686 | 21.856 | 23.298 | 24.399 | 25.298 | 26.060 | 26.723 | 27.310 | 27.838 |
| 0.750 | 20.265 | 22.492 | 23.971 | 25.103 | 26.027 | 26.810 | 27.492 | 28.079 | 28.640 |
| 0.775 | 20.841 | 23.124 | 24.641 | 25.803 | 26.751 | 27.556 | 28.257 | 28.846 | 29.438 |
| 0.800 | 21.414 | 23.753 | 25.308 | 26.499 | 27.472 | 28.190 | 29.018 | 29.657 | 30.231 |
| 0.825 | 21.985 | 24.379 | 25.972 | 27.192 | 28.190 | 29.037 | 29.775 | 30.450 | 31.102 |
| 0.850 | 22.554 | 25.003 | 26.633 | 27.882 | 28.904 | 29.772 | 30.529 | 31.200 | 31.805 |
| 0.875 | 23.121 | 25.625 | 27.292 | 28.569 | 29.615 | 30.504 | 31.279 | 31.967 | 32.586 |
| 0.900 | 23.686 | 26.244 | 27.947 | 29.253 | 30.323 | 31.232 | 32.025 | 32.739 | 33.364 |
| 0.925 | 24.249 | 26.860 | 28.600 | 29.934 | 31.027 | 31.957 | 32.768 | 33.489 | 34.138 |
| 0.950 | 24.810 | 27.474 | 29.250 | 30.613 | 31.729 | 32.679 | 33.508 | 34.245 | 34.908 |
| 0.975 | 25.369 | 28.086 | 29.898 | 31.288 | 32.428 | 33.398 | 34.245 | 34.997 | 35.676 |
| 1.000 | 25.927 | 28.696 | 30.543 | 31.962 | 33.124 | 34.114 | 34.979 | 35.747 | 36.440 |
| 1.025 | 26.482 | 29.304 | 31.186 | 32.632 | 33.818 | 34.827 | 35.709 | 36.494 | 37.201 |
| 1.050 | 27.036 | 29.910 | 31.827 | 33.300 | 34.509 | 35.538 | 36.437 | 37.237 | 37.959 |
| 1.075 | 27.588 | 30.514 | 32.466 | 33.966 | 35.197 | 36.246 | 37.163 | 37.978 | 38.716 |
| 1.100 | 28.139 | 31.116 | 33.103 | 34.630 | 35.883 | 36.951 | 37.885 | 38.716 | 39.466 |
| 1.125 | 28.688 | 31.716 | 33.737 | 35.291 | 36.567 | 37.654 | 38.605 | 39.452 | 40.215 |
| 1.150 | 29.236 | 32.315 | 34.370 | 35.950 | 37.248 | 38.355 | 39.323 | 40.185 | 40.962 |
| 1.175 | 29.782 | 32.911 | 35.001 | 36.608 | 37.927 | 39.053 | 40.038 | 40.915 | 41.706 |
| 1.200 | 30.327 | 33.506 | 35.629 | 37.263 | 38.604 | 39.749 | 40.751 | 41.643 | 42.448 |
| 1.225 | 30.871 | 34.100 | 36.256 | 37.916 | 39.279 | 40.443 | 41.461 | 42.368 | 43.187 |
| 1.250 | 31.413 | 34.692 | 36.882 | 38.567 | 39.952 | 41.134 | 42.169 | 43.091 | 43.924 |
| 1.275 | 31.953 | 35.282 | 37.505 | 39.216 | 40.623 | 41.824 | 42.875 | 43.812 | 44.658 |
| 1.300 | 32.493 | 35.871 | 38.127 | 39.864 | 41.292 | 42.511 | 43.579 | 44.531 | 45.390 |
| 1.325 | 33.031 | 36.458 | 38.747 | 40.510 | 41.959 | 43.197 | 44.281 | 45.247 | 46.120 |
| 1.350 | 33.568 | 37.044 | 39.365 | 41.154 | 42.624 | 43.880 | 44.980 | 45.961 | 46.848 |
| 1.375 | 34.104 | 37.628 | 39.982 | 41.796 | 43.287 | 44.562 | 45.678 | 46.674 | 47.574 |
| 1.400 | 34.639 | 38.211 | 40.598 | 42.436 | 43.949 | 45.241 | 46.374 | 47.384 | 48.299 |
| 1.425 | 35.172 | 38.792 | 41.212 | 43.075 | 44.609 | 45.919 | 47.068 | 48.092 | 49.019 |
| 1.450 | 35.705 | 39.373 | 41.824 | 43.713 | 45.267 | 46.595 | 47.760 | 48.799 | 49.738 |
| 1.475 | 36.236 | 39.952 | 42.435 | 44.349 | 45.923 | 47.270 | 48.450 | 49.503 | 50.456 |
| 1.500 | 36.766 | 40.530 | 43.045 | 44.983 | 46.578 | 47.942 | 49.138 | 50.206 | 51.171 |
| 1.525 | 37.295 | 41.116 | 43.656 | 45.645 | 47.245 | 48.581 | 49.826 | 50.911 | 51.881 |
| 1.550 | 37.823 | 41.701 | 44.271 | 46.316 | 47.548 | 49.181 | 50.519 | 51.604 | 52.584 |
| 1.575 | 38.351 | 42.286 | 44.882 | 46.991 | 47.851 | 49.536 | 51.212 | 52.299 | 53.287 |
| 1.600 | 38.878 | 42.871 | 45.497 | 47.666 | 48.154 | 49.841 | 51.919 | 53.009 | 53.991 |
| 1.625 | 39.405 | 43.456 | 46.112 | 48.281 | 48.457 | 50.146 | 52.626 | 53.713 | 54.694 |
| 1.650 | 39.932 | 44.035 | 46.729 | 48.906 | 48.762 | 50.471 | 53.333 | 54.419 | 55.397 |
| 1.675 | 40.459 | 44.614 | 47.351 | 49.521 | 49.067 | 50.796 | 54.040 | 55.122 | 56.100 |
| 1.700 | 40.986 | 45.193 | 47.974 | 50.136 | 49.362 | 51.121 | 54.745 | 55.825 | 56.803 |
| 1.725 | 41.513 | 45.772 | 48.597 | 50.751 | 49.658 | 51.446 | 55.446 | 56.528 | 57.506 |
| 1.750 | 42.040 | 46.351 | 49.220 | 51.366 | 49.953 | 51.771 | 56.151 | 57.231 | 58.209 |
| 1.775 | 42.567 | 46.929 | 49.849 | 51.981 | 50.248 | 52.096 | 56.856 | 57.936 | 58.912 |
| 1.800 | 43.094 | 47.508 | 50.478 | 52.596 | 50.543 | 52.421 | 57.561 | 58.641 | 59.615 |
| 1.825 | 43.621 | 48.087 | 51.107 | 53.205 | 50.838 | 52.746 | 58.266 | 59.346 | 60.318 |
| 1.850 | 44.148 | 48.666 | 51.736 | 53.814 | 51.133 | 53.071 | 58.971 | 60.051 | 61.021 |
| 1.875 | 44.675 | 49.245 | 52.365 | 54.423 | 51.428 | 53.396 | 59.676 | 60.756 | 61.724 |
| 1.900 | 45.202 | 49.824 | 52.994 | 55.032 | 51.723 | 53.721 | 60.381 | 61.459 | 62.427 |
| 1.925 | 45.729 | 50.403 | 53.623 | 55.641 | 52.018 | 54.046 | 61.086 | 62.162 | 63.130 |
| 1.950 | 46.256 | 50.982 | 54.252 | 56.250 | 52.313 | 54.371 | 61.791 | 62.865 | 63.833 |
| 1.975 | 46.783 | 51.561 | 54.881 | 56.859 | 52.608 | 54.696 | 62.496 | 63.568 | 64.536 |
| 2.000 | 47.310 | 52.140 | 55.510 | 57.468 | 52.903 | 55.021 | 63.201 | 64.271 | 65.239 |
| 2.025 | 47.837 | 52.719 | 56.139 | 58.077 | 53.198 | 55.346 | 63.906 | 64.976 | 65.942 |
| 2.050 | 48.364 | 53.298 | 56.768 | 58.686 | 53.493 | 55.671 | 64.611 | 65.681 | 66.645 |
| 2.075 | 48.891 | 53.877 | 57.397 | 59.295 | 53.788 | 56.000 | 65.316 | 66.386 | 67.348 |
| 2.100 | 49.418 | 54.456 | 58.026 | 59.904 | 54.083 | 56.325 | 66.021 | 67.091 | 68.051 |
| 2.125 | 49.945 | 55.035 | 58.655 | 60.513 | 54.378 | 56.650 | 66.726 | 67.796 | 68.754 |
| 2.150 | 50.472 | 55.614 | 59.284 | 61.122 | 54.673 | 56.975 | 67.431 | 68.501 | 69.457 |
| 2.175 | 51.000 | 56.193 | 59.893 | 61.731 | 54.968 | 57.300 | 68.136 | 69.206 | 70.160 |
| 2.200 | 51.527 | 56.772 | 60.502 | 62.340 | 55.263 | 57.625 | 68.841 | 69.911 | 70.863 |
| 2.225 | 52.054 | 57.351 | 61.111 | 62.949 | 55.558 | 57.950 | 69.546 | 70.616 | 71.566 |
| 2.250 | 52.581 | 57.930 | 61.720 | 63.558 | 55.853 | 58.275 | 70.251 | 71.321 | 72.269 |
| 2.275 | 53.108 | 58.509 | 62.329 | 64.167 | 56.148 | 58.600 | 70.956 | 72.026 | 72.972 |
| 2.300 | 53.635 | 59.088 | 62.938 | 64.776 | 56.443 | 58.925 | 71.661 | 72.731 | 73.675 |
| 2.325 | 54.162 | 59.667 | 63.547 | 65.385 | 56.738 | 59.250 | 72.366 | 73.436 | 74.378 |
| 2.350 | 54.689 | 60.246 | 64.156 | 65.994 | 57.033 | 59.575 | 73.071 | 74.141 | 75.081 |
| 2.375 | 55.216 | 60.825 | 64.765 | 66.603 | 57.328 | 59.900 | 73.776 | 74.846 | 75.784 |
| 2.400 | 55.743 | 61.404 | 65.374 | 67.212 | 57.623 | 60.225 | 74.481 | 75.549 | 76.487 |
| 2.425 | 56.270 | 61.983 | 65.983 | 67.821 | 57.918 | 60.550 | 75.186 | 76.252 | 77.190 |
| 2.450 | 56.797 | 62.562 | 66.592 | 68.430 | 58.213 | 60.875 | 75.891 | 76.955 | 77.893 |
| 2.475 | 57.324 | 63.141 | 67.201 | 69.039 | 58.508 | 61.200 | 76.596 | 77.658 | 78.596 |
| 2.500 | 57.851 | 63.720 | 67.810 | 69.648 | 58.803 | 61.525 | 77.301 | 78.361 | 79.299 |
| 2.525 | 58.378 | 64.299 | 68.419 | 70.257 | 59.098 | 61.850 | 78.006 | 79.064 | 79.999 |
| 2.550 | 58.905 | 64.878 | 69.028 | 70.866 | 59.393 | 62.175 | 78.711 | 79.767 | 80.699 |
| 2.575 | 59.432 | 65.457 | 69.637 | 71.475 | 59.688 | 62.500 | 79.416 | 80.470 | 81.399 |
| 2.600 | 59.959 | 66.036 | 70.246 | 72.084 | 59.983 | 62.825 | 80.121 | 81.173 | 82.099 |
| 2.625 | 60.486 | 66.615 | 70.855 | 72.693 | 60.278 | 63.150 | 80.826 | 81.876 | 82.799 |
| 2.650 | 61.013 | 67.194 | 71.464 | 73.302 | 60.573 | 63.475 | 81.531 | 82.579 | 83.499 |
| 2.675 | 61.540 | 67.773 | 72.073 | 73.911 | 60.868 | 63.800 | 82.236 | 83.282 | 84.199 |
| 2.700 | 62.067 | 68.352 | 72.682 | 74.520 | 61.163 | 64.125 | 82.941 | 83.985 | 84.899 |
| 2.725 | 62.594 | 68.931 | 73.291 | 75.129 | 61.458 | 64.450 | 83.646 | 84.688 | 85.599 |
| 2.750 | 63.121 | 69.510 | 73.900 | 75.738 | 61.753 | 64.775 | 84.351 | 85.391 | 86.299 |