Capacitance Calculations

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***Abstract*—This document describes calculation methods for distributed capacitances of objects with several particular shapes, and methods for the evaluation of the electric fields and forces. It's fundamentally a collection of formulas, some not very easy to find in the literature. The algorithms were implemented in the Inca program, available at http://www.coe.ufrj.br/~acmq/programs.**

I. Introduction

Most of the formulas below are known since long time, most dating from works in the XIX century. Some appear in Maxwell's book [1], and some in other collections of explicit formulas for electromagnetic problems, as [2], or in other early works as [3]-[5]. In most cases I have just adapted the notation, but some derivations not found in other works are presented too.

In most of the early works, capacitance is expressed in units of length. For example, the capacitance of a sphere of radius a in free space is listed in [1] and [2] as $C=a$. To convert this unit to Farads, it's necessary to multiply the value by $4\pi\varepsilon_0$, where ε_0 is the permissivity of vacuum, $\varepsilon_0 = 8.8541878 \times 10^{-12}$. ε_0 can be calculated from the speed of light c and from the magnetic permeability of vacuum, $\mu_0 = 4\pi \times 10^{-7}$ (a definition), from the relation:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \quad (1)$$

The capacitance of a sphere of radius a meters is then:

$$C_{\text{sphere}} = 4\pi\varepsilon_0 a = 111.26501a \text{ pF} \quad (2)$$

Other figures that have simple expressions for the free-space capacitance are:

A thin flat disk with radius a [2]:

$$C_{\text{disk}} = 8\varepsilon_0 a = 70.833503a \text{ pF} \quad (3)$$

An open hemisphere with radius a [2]:

$$C_{\text{open hemisphere}} = 4\pi\varepsilon_0 a(1/2+1/\pi) = 91.049254a \text{ pF} \quad (4)$$

A closed (with a flat disk) hemisphere with radius a [2]:

$$C_{\text{closed hemisphere}} = 8\pi\varepsilon_0 a\left(1-1/\sqrt{3}\right) = 94.052249 \text{ pF} \quad (5)$$

Two spheres with radius a in contact [1]:

$$C_{\text{two spheres}} = 8\pi\varepsilon_0 a\text{Ln}(2) = 154.24505a \text{ pF} \quad (6)$$

An "oblate spheroid" is the figure generated by the rotation of an ellipse around its minor axis. A "prolate spheroid" is generated by the rotation of an ellipse around its major axis. The capacitances of these figures are, considering the major axis with length $2a$ and the minor axis with the length $2b$ [2]:

$$C_{\text{oblate}} = 4\pi\varepsilon_0 \frac{\sqrt{a^2-b^2}}{\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}} \quad (7)$$

$$C_{\text{prolate}} = 4\pi\varepsilon_0 \frac{\sqrt{a^2-b^2}}{\ln\frac{a+\sqrt{a^2-b^2}}{b}} \quad (8)$$

Note the limits when $a = b$ reducing to (2), and the reduction to (3) when $b = 0$ in (7).

For bodies embedded in materials with other permissivities, it's just a question of multiplying ε_0 by the relative permissivity ε of the material. The case when different dielectrics are present on the structure will be not discussed here.

II. Capacitance of a Toroid

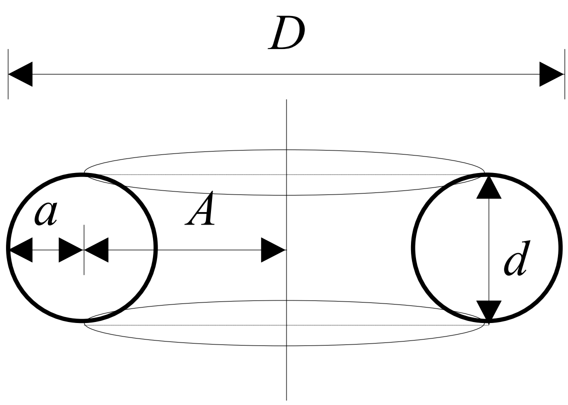

Fig. 1. Toroid with major diameter D, minor diameter d, center radius A, and tube radius a.

From [2] (the same formula appears in [3], that is probably the origin of this formula, but in a somewhat different notation) the capacitance of a toroid with major diameter D and minor diameter d, $d<D/2$, (fig. 1) is:

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$$C = 16\varepsilon_0 \sqrt{A^2 - a^2} \sum_{n=0}^{\infty} \sigma_n \frac{Q_{n-1/2}(x)}{P_{n-1/2}(x)};$$
$$\sigma_n = 1/2 \text{ for } n = 0, 1 \text{ for } n > 0; \quad (9)$$
$$A = \frac{D-d}{2};\;\; a = \frac{d}{2};\;\; x = \frac{A}{a}$$

where $P_{n-1/2}(x)$ and $Q_{n-1/2}(x)$ are Legendre functions, or in this case, “toroidal functions”.

These functions can be evaluated in the following way: The first two terms can be obtained from their relations with the complete elliptic integrals of first and second kinds:

$$Q_{-1/2}(x) = kK;$$
$$Q_{1/2}(x) = 2\frac{K-E}{k} - kK; \quad (10)$$
$$P_{-1/2}(x) = \frac{2}{\pi}kK';$$
$$P_{1/2}(x) = \frac{2}{\pi}\left(\frac{2E'}{k} - kK'\right)$$

The modulus for the elliptic integrals *K* and *E* is:

$$k = \sqrt{\frac{2a}{A+a}} \quad (11)$$

And for the elliptic integrals *K*’ and *E*’ (evaluated in the same way, with modulus *k*’):

$$k' = \sqrt{1-k^2} \quad (12)$$

This is enough for the evaluation of the two first terms of the series (enough for thin toroids). The other terms can be obtained using the recursion for Legendre functions, identical for both functions:

$$P_{m+1/2}(x) = \frac{2mxP_{m-1/2}(x) - (m-1/2)P_{m-1/2-1}(x)}{m+1/2};$$
$$Q_{m+1/2}(x) = \frac{2mxQ_{m-1/2}(x) - (m-1/2)Q_{m-1/2-1}(x)}{m+1/2} \quad (13)$$

where $m=n-1$. All the terms can then be easily computed, starting with $n=2$ in the series (9), or $m=1$.

The complete elliptic integrals are the irreducible functions:

$$K = \mathrm{F}(k, \pi/2) = \mathrm{F}(k) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \quad (14)$$
$$E = \mathrm{E}(k, \pi/2) = \mathrm{E}(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

They can be quickly and precisely evaluated using the arithmetic-geometric mean method, below implemented in a Pascal routine:

```
{
Complete elliptic integrals of first
and second classes - AGM method.
Returns the global variables:
Ek=E(c) and Fk=F(c) (E and K)
Doesn’t require more than 7 iterations for
c between 0 and 0.9999999999.
Reference: Pi and the AGM, J. Borwein and
P. Borwein, John Wiley & Sons.
}
procedure EF(c:real);
var
  a,b,a1,b1,E,i:real;
begin
  a:=1;
  b:=sqrt(1-sqr(c));
  E:=1-sqr(c)/2;
  i:=1;
  repeat
    a1:=(a+b)/2;
    b1:=sqrt(a*b);
    E:=E-i*sqr((a-b)/2);
    i:=2*i;
    a:=a1;
    b:=b1;
  until abs(a-b)<1e-15;
  Fk:=pi/(2*a);
  Ek:=E*Fk
end;
```

III. Approximate calculations for partial toroids

A partial toroid can be described as a surface generated by the revolution of a partial circle of radius *a* centered at a distance *A* along the radial axis *r* from the revolution axis *z*. The circle limits are defined by two angles θ_1 and θ_2. See fig. 2.

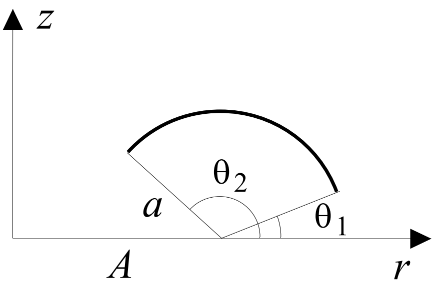

Fig. 2: A partial toroidal surface is generated by the rotation of a partial circle around the vertical axis.

With this formulation several figures can be generated, as a regular toroid when θ_2-θ_1 = 2π and $a<A$, a sphere when $A = 0$, $\theta_2 = -\theta_1 = \pi/2$, an open hemisphere, etc. Even overlapping toroids, with $a>A$, can be generated.

This surface can be decomposed in a set of *n* infinitely thin circles with axles at the *z* axis, positioned at heights z_i, and with radii r_i, uniformly spaced at angles $\Delta\theta$ along the surface:

$$\Delta\theta = \frac{\theta_2 - \theta_1}{n}$$
$$\theta_i = \theta_1 + \frac{\Delta\theta}{2} + (i-1)\Delta\theta,\;\; i = 1...n \quad (15)$$
$$r_i = A + a\cos\theta_i$$
$$z_i = a\sin\theta_i$$

Each of these rings has a uniform charge distribution, with a total charge q_i. The potential Ψ due to each ring *i* at any given position r_0, *z0* is given by:

$$\Psi_i(r_0,z_0)=\frac{q_i}{4\pi^2\varepsilon_0\sqrt{|r_0 r_i|}}Q_{-1/2}\left(\frac{2}{k^2}-1\right)=\frac{q_i}{2\pi^2\varepsilon_0 R_1}K$$
$$k=\frac{2\sqrt{|r_0 r_i|}}{R_1} \quad (16)$$
$$R_1=\sqrt{(|r_i|+|r_0|)^2+(z_0-z_i)^2}$$

The absolute values allow correct treatment of the cases when some radii are negative.

Considering then the mutual influences among all the rings, a matrix **P** can be computed, that allows the calculation of the potentials v_i at each ring, once the charges q_i are known [1]:

$$\mathbf{v}=\mathbf{Pq}$$
$$P_{ij}=P_{ji}=\Psi_i(r_j,z_j)/q_i \quad (17)$$
$$P_{ii}=\Psi_i(r_i,z_i+R)/q_i$$

For the calculation of the "self-potentials" P_{ii}, something must be assumed about the radius of the rings, R. The formulation calculates then the potentials at a distance R above the rings. The maximum physically possible value of R would be when adjacent rings touch:

$$R_{max}=a\sin\frac{\Delta\theta}{2} \quad (18)$$

Any reasonable fraction of this value can be used with similar results, but there is one that produces better results in the next calculation, that was found (by trying!) to, curiously, be:

$$R=\frac{a}{\pi}\sin\frac{\Delta\theta}{2} \quad (19)$$

This radius makes the area of the surface of the ring to be identical to the flat area represented by it, at least in the cases when $\theta = n\pi/2$ and small $\Delta\theta$ (as in the equator and poles of a sphere split in many rings).

The charge distribution for uniform potential V at all the rings can be calculated by inverting the matrix **P**. The total charge in each ring is then obtained from a sum of the corresponding lines of the inverse of **P**, **C**. The coefficients of **C** are the influence coefficients k_{ij}:

$$q_i=V\sum_{j=1}^{n}k_{ij} \quad (20)$$
$$\mathbf{C}=\mathbf{P}^{-1}$$

And the capacitance of the whole assembly is simply the sum of all the elements of **C**:

$$C_{total}=\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij} \quad (21)$$

Surface electric field

The electric field at any point of the surface is normal to it and can be calculated by Gauss' law as proportional to the charge density at that point of the surface:

$$E_i=\frac{\rho_i}{\varepsilon_0} \quad (22)$$

where ρ_i is the surface charge density, uniform around the ring i. For a closed surface, the electric field is entirely at the outer surface. In this case, it can be calculated directly from the charge distribution alone.

Assuming constant voltage at the surface of the object, the charges at the rings can be calculated by (20). The ring i has a length $2\pi r_i$ and a total charge q_i. The ring represents a thin belt with width equal to $a\Delta\theta$. The charge density and the electric field in a small length l are then:

$$\rho_i=\frac{q_i}{2\pi r_i}l\frac{1}{la\Delta\theta}=\frac{q_i}{2\pi r_i a\Delta\theta} \quad (23)$$
$$E_i=\frac{q_i}{2\pi r_i a\Delta\theta\varepsilon_0}$$

An important application of this calculation is the determination of the breakout voltage of the object, the voltage that causes ionization of the air around it when the electric field reaches about 3 MV/m:

$$V_{max}=3000/\mathrm{Max}\,E_i\ \text{kV} \quad (24)$$

For a toroid, this value occurs at the maximum diameter. In the case of open objects, it's not possible to calculate the surface electric field in this way, because it is split in an unknown way between the two sides of the surface. The calculation is also meaningless if the object has an edge, where the electric field is ideally infinite. A strange problem with (23) is that it fails when the rings are close to the center of a spherical surface. The last ring appears to have significantly less charge than it should have (around 92%). The calculations for capacitance, however, continue to result in good values.

IV. General truncated cones

Any other figure with circular symmetry can be analyzed by the same method. A simple case is the revolution of a straight line around the central axle, that generates figures ranging from a flat disk with a possible central hole to a cone or an open cylinder.

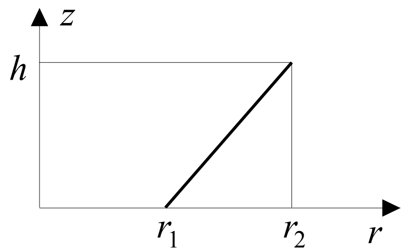

Fig. 3. A line that rotates around the vertical axis.

The coordinates or the rings are then:

$$\Delta z = \frac{h}{n}$$
$$\Delta r = \frac{r_2 - r_1}{n} \quad (25)$$
$$z_i = \frac{\Delta z}{2} + (i-1)\Delta z,\ \ i = 1...n$$
$$r_i = r_1 + \frac{\Delta r}{2} + (n-1)\Delta r,\ \ i = 1...n$$

The radius to use in the calculation of the self-potentials would be, still using the maximum divided by π:

$$R = \frac{\sqrt{(r_2 - r_1)^2 + h^2}}{2n\pi} \quad (26)$$

With this radius, the surface charge density and the surface electric field (for a closed object) can be calculated considering that the surface area of the ring is identical to the belt area represented by it, what is approximately valid also for the case or curves, using R given by (19):

$$\rho_i = \frac{q_i}{2\pi r_i} l \frac{1}{l 2\pi R} = \frac{q_i}{4\pi^2 r_i R}$$
$$E_i = \frac{q_i}{4\pi^2 r_i R \varepsilon_0} \quad (27)$$

V. ELECTRIC FIELD FROM A RING

The electric field anywhere can be calculated by adding the electric fields due to the rings. From (16), the radial and axial components of the electric field can be calculated by differentiation, resulting in:

$$E_{radial} = -\frac{d\Psi_i}{dr_0} = -\frac{q_i \operatorname{sign} r_i}{2\pi^2 \varepsilon_0 R_1^3} \left(\begin{array}{l} -(|r_i| - |r_0|)K + \\ + \frac{E - k'^2 K}{k k'^2} \left(\frac{2|r_0|}{k} - k(|r_i| - |r_0|) \right) \end{array} \right) \quad (28)$$

$$E_{axial} = -\frac{d\Psi_i}{dz_0} = \frac{q_i (z_i - z_0) E}{2\pi^2 \varepsilon_0 R_1^3 k'^2} \quad (29)$$

$$E_{total} = \sqrt{E_{radial}^2 + E_{axial}^2} \quad (30)$$

where the derivative of the elliptic integral K in relation to the modulus k was used (the derivative of E is listed below too for reference, but was not necessary):

$$\frac{dK}{dk} = \frac{E - k'^2 K}{k k'^2};\ \ \frac{dE}{dk} = \frac{E - K}{k}$$
$$k'^2 = 1 - k^2 \quad (31)$$

VI. GENERAL CASE WITH AXIAL SYMMETRY

The capacitance matrix and the potential and electric field around a series of objects with axial symmetry decomposed in thin rings can then be easily calculated. The objects are decomposed in series of partial toroids conical sheets, and other shapes (as ellipses) and these parts are decomposed in rings. To obtain the capacitance matrix, it's just a question of adding the terms of the total capacitance matrix that correspond to the rings that belong to the objects, instead of adding them all to obtain the capacitance of the entire object. The charges in all the rings can be obtained from the complete equation **q**=**CV**, with the assigned voltages in the objects arranged in **V** in correspondence with the rings that belong to the objects. The potential anywhere around the objects is obtained by adding (16) for all the rings, and the electric field by adding (28) and (29) and using (30). The terms at the diagonal of the capacitance matrix correspond to the capacitances of the objects to ground when all the other objects are grounded too. The influence coefficients out of the diagonal measure the relation between the charge induced in one object and the voltage in another, when all the other objects are grounded. From the capacitance matrix, a model of the circuit using lumped capacitors can be derived, by observing the equivalence:

$$\mathbf{C} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{12} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1n} & k_{2n} & \cdots & k_{nn} \end{bmatrix} = \begin{bmatrix} C_1 + C_{12} + ... + C_{1n} & -C_{12} & \cdots & -C_{1n} \\ -C_{12} & C_2 + C_{12} + ... + C_{2n} & \cdots & -C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -C_{1n} & -C_{2n} & \cdots & C_n + C_{nn-1} + ... + C_{1n} \end{bmatrix} \quad (32)$$

C_1, C_2, …, C_n are direct capacitances between the elements and the ground, and the other elements are the negative of the floating capacitances between the objects. The direct capacitance to ground for the object i is just the sum of the elements in the line, or column, i of **C**.

VII. MAXIMUM ELECTRIC FIELD BETWEEN TWO SPHERES

A good test for these field calculations is the known formula for the maximum electric field between two different spheres [4]. The expression comes directly from the method of images developed by Lord Kelvin [10]. For two spheres of radii a and b, $a<b$, with distance between centers c, at potentials v_1 and v_2, the maximum electric field at the surface of the smaller sphere (assumed as being where the surface of the smaller sphere intercepts the line between the centers of the spheres) is given by:

$$E_{\max} = \frac{(1+\xi)^2}{a(1-\xi)} \left[\begin{array}{l} v_1 \left\{ \frac{1-\xi}{(1+\xi)^2} + \alpha \frac{1-\xi\alpha^2}{(1+\xi\alpha^2)^2} + \alpha^2 \frac{1-\xi\alpha^4}{(1+\xi\alpha^4)^2} + \cdots \right\} \\ -v_2 \left\{ \eta \frac{1-\eta\alpha}{(1+\eta\alpha)^2} + \alpha\eta \frac{1-\eta\alpha^3}{(1+\eta\alpha^3)^2} + \alpha^2 \eta \frac{1-\eta\alpha^5}{(1+\eta\alpha^5)^2} + \ldots \right\} \end{array} \right]; \quad (33)$$
$$\xi = \frac{a + b\alpha}{c};\ \ \eta = \frac{b + a\alpha}{c};$$
$$\alpha = \text{root} < 1 \text{ of } (a\alpha + b)(b\alpha + a) = c^2 \alpha$$

This formula converges slowly when the spheres are at small distance, but the speed is acceptable. [4]

develops a better expression for the case of spheres at small distance too. The choice of α appears to work correctly too when the other root is used. The formula works for any choice of v_1 and v_2, but always calculates the electric field at the point of the sphere with radius a closest to the other sphere, even when this is not the point of maximum electric field (as when v_1 and v_2 have the same sign). Note that at that point the calculation using rings, as formulated, calculates an electric field slightly smaller than the correct value.

VIII. Capacitances of two spheres

Similar formulas, due to Kirchhoff, lead to the capacitance matrix of two spheres [5]. For two spheres with radii a and b and distance between centers c:

$$\begin{aligned}
k_{11} &= 8\pi\varepsilon_0\lambda\left[\frac{\xi}{1+\xi^2}+\frac{\alpha\xi}{1+\alpha^2\xi^2}+\frac{\alpha^2\xi}{1+\alpha^4\xi^2}+\cdots\right];\\
k_{22} &= 8\pi\varepsilon_0\lambda\left[\frac{\eta}{1+\eta^2}+\frac{\alpha\eta}{1+\alpha^2\eta^2}+\frac{\alpha^2\eta}{1+\alpha^4\eta^2}+\cdots\right];\\
k_{12} &= -8\pi\varepsilon_0\lambda\left[\frac{\alpha}{1+\alpha^2}+\frac{\alpha^2}{1+\alpha^4}+\frac{\alpha^3}{1+\alpha^6}+\cdots\right];\\
\xi &= \sqrt{1+\frac{\lambda^2}{a^2}}-\frac{\lambda}{a};\quad \alpha=\frac{c\xi-a}{b};\quad \eta=\frac{\alpha}{\xi};\\
\lambda &= \frac{\sqrt{(c+a+b)(c-a-b)(c+a-b)(c-a+b)}}{2c}
\end{aligned} \tag{34}$$

These formulas also converge slowly when the spheres are at small distance. [5] shows better formulas for small distances. [11][12] have the exact solution when the spheres are touching, eq. (49).

The coefficients of the capacitance matrix represent the ratio between the induced charges and the voltages. k_{11} and k_{12} represent capacitances to ground from a sphere with the other sphere grounded, and $-k_{12}$ is the floating capacitance between the spheres. The differential capacitance between the spheres is obtained by assuming opposite charges $\pm q$ on them:

$$C_{diff}=\frac{v_1-v_2}{q}=\frac{k_{11}k_{22}-k_{12}^{\ 2}}{k_{11}+k_{22}+2k_{12}} \tag{35}$$

The capacitances to ground with the other sphere floating can also be calculated, by assuming zero charge in the floating sphere:

$$C_{10}=\frac{k_{11}k_{22}-k_{12}^{\ 2}}{k_{22}};\quad C_{20}=\frac{k_{11}k_{22}-k_{12}^{\ 2}}{k_{11}} \tag{36}$$

The capacitance of both spheres to ground is simply the sum of the four coefficients, because both are at the same potential and the total charge appears in the ratio $C = Q/V$. This capacitance can be used to find the relation between total charge and voltage in a pith-ball electroscope. For equal spheres it varies between (6) when the spheres are touching and two times (2) when they are far apart:

$$C_{pair}=k_{11}+k_{22}+2k_{12} \tag{37}$$

IX. Potential and electric field around a toroid

The solution of this problem can be traced to [3]. The formula for the potential also appears in [2]. The potential around an isolated toroid in free space, with central radius A and tube radius a, at a radial distance r and axial distance z from the center, is found as:

$$\begin{aligned}
&\Psi(\alpha,\beta)=\frac{2V}{\pi}\sqrt{2(\cosh\beta-\cos\alpha)}\sum_{n=0}^{\infty}\sigma_n\frac{Q_{n-1/2}(x)}{P_{n-1/2}(x)}P_{n-1/2}(\cosh\beta)\cos n\alpha;\\
&\sigma_n=1/2 \text{ for } n=0, 1 \text{ for } n>0;\\
&x=\frac{A}{a};\quad c=\sqrt{A^2-a^2};\\
&\beta=\frac{1}{2}\ln\frac{z^2+(r+c)^2}{z^2+(r-c)^2};\\
&\sin\alpha=\frac{z}{r}\sinh\beta;\quad \cos\alpha=\cosh\beta-\frac{c}{r}\sinh\beta\\
&\alpha=\tan^{-1}\frac{\sin\alpha}{\cos\alpha}=\tan^{-1}\frac{2cz}{r^2+z^2-c^2};
\end{aligned} \tag{38}$$

The surface electric field can be found by the differentiation of (38). The maximum occurs when cosh β = x =A/a (toroid surface) and α=0 (major diameter). The result, hinted in [3] but not developed, is the series:

$$E=\frac{4\sqrt{2}V(x-\cos\alpha)^{3/2}}{\pi d(x^2-1)}\sum_{n=0}^{\infty}\sigma_n\frac{\cos n\alpha}{P_{n-1/2}(x)} \tag{39}$$

The ideal exact breakdown voltage can then be obtained as in (24). This series converges somewhat more slowly than (9) but still can achieve high precision. The series (38) may lose precision due to errors in the evaluation of $Q_{n+1/2}(x)$ by the recursion (13).

X. Other capacitances

Other capacitances, of shapes without circular symmetry, are known from numerical analysis. Classical cases are the capacitances of the square [13] and triangular [6] flat plates, the cube [7] and the tetrahedron [6], all with side a:

$$C_{\text{square}} = 4\pi\varepsilon_0 a \times 0.3667896 = 40.81085a \text{ pF} \tag{40}$$

$$C_{\text{triangle}} = 4\pi\varepsilon_0 a \times 0.25096 = 27.293a \text{ pF} \tag{41}$$

$$C_{\text{cube}} = 4\pi\varepsilon_0 a \times 0.6606782 = 73.51036a \text{ pF} \tag{42}$$

$$C_{\text{tetrahedron}} = 4\pi\varepsilon_0 a \times 0.35688 = 39.708a \text{ pF} \tag{43}$$

XI. Forces between rings

The force between thin coaxial filaments can be calculated by an integration of the Coulomb force between them. For two rings with radii A and a and separation b, containing charges q_1 and q_2, the force is given by the double integral (44). The term cosθ projects the force along the axis:

$$F = \frac{1}{4\pi\varepsilon_0}\iint \frac{\cos\theta}{r^2}\,dq\,dq';$$
$$r = \sqrt{A^2 + a^2 + b^2 - 2Aa\cos(\varphi - \varphi')};$$
$$\cos\theta = b/r; \qquad (44)$$
$$dq = \frac{q_1}{2\pi}\,d\varphi;$$
$$dq' = \frac{q_2}{2\pi}\,d\varphi';$$
$$F_{12} = \frac{q_1 q_2}{16\pi^3\varepsilon_0}\int_0^{2\pi}\int_0^{2\pi}\frac{b\,d\varphi\,d\varphi'}{\left(\sqrt{A^2 + a^2 + b^2 - 2Aa\cos(\varphi - \varphi')}\right)^3}$$

This integral can be exactly solved in terms of the complete elliptic integral of the third kind (45), which is also easy to evaluate. The code below computes in few iterations the three complete elliptic integrals $E(k)$, $F(k)$, and $\Pi(k,c)$:

$$\Pi(k,c) = \int_0^{\pi/2}\frac{d\varphi}{\left(1 - c^2\sin^2\varphi\right)\sqrt{1 - k^2\sin^2\varphi}} \qquad (45)$$

```
{
Complete elliptic integrals of first, second,
and third kinds - AGM
Returns the global variables Ek=E(k), Fk=F(k),
and IIkc=II(k,c)
Reference: Garrett, Journal of Applied
Physics, 34, 9, 1963, p. 2571
}
procedure EFII(k,c:real);
var
  a,b,d,e,f,a1,b1,d1,e1,f1,S,i:real;
begin
  a:=1;
  b:=sqrt(1-sqr(k));
  d:=(1-sqr(c))/b;
  e:=sqr(c)/(1-sqr(c));
  f:=0;
  i:=1/2;
  S:=i*sqr(a-b);
  repeat
    a1:=(a+b)/2;
    b1:=sqrt(a*b);
    i:=2*i;
    S:=S+i*sqr(a1-b1);
    d1:=b1/(4*a1)*(2+d+1/d);
    e1:=(d*e+f)/(1+d);
    f1:=(e+f)/2;
    a:=a1;
    b:=b1;
    d:=d1;
    e:=e1;
    f:=f1;
  until (abs(a-b)<1e-15) and (abs(d-1)<1e-15);
  Fk:=pi/(2*a);
  Ek:=Fk-Fk*(sqr(k)+S)/2;
  IIkc:=Fk*f+Fk;
end;
```

The solution for the force is:

$$F_{12} = \frac{q_1 q_2 b}{2\pi^2\varepsilon_0 r_1^3}\Pi(k,c)$$
$$k = c = \frac{2\sqrt{Aa}}{r_1} \qquad (46)$$
$$r_1 = \sqrt{(A+a)^2 + b^2}$$

The formula reduces to the well-known case of a charged ring and a point charge if one ring has zero radius. In this case $k = c = 0$ and $\Pi(0,0) = \pi/2$:

$$F_{12} = \frac{q_1 q_2 b}{2\pi^2\varepsilon_0\left(A^2 + b^2\right)^{3/2}}\frac{\pi}{2} = \frac{q_1 q_2 b}{4\pi\varepsilon_0\left(A^2 + b^2\right)^{3/2}} \qquad (47)$$

To calculate the forces in conductors decomposed in rings, the charges in the rings are first calculated using **q**=**CV** and then the forces in each ring can be obtained by (46), by adding the values obtained between a given ring and all the others. Finally, the forces in all the rings belonging to each conductor are added. The calculations for rings belonging to a single conductor can be omitted, because they add to zero (and this is a good test of the algorithm).

The same result can be obtained by computing the forces multiplying the total axial electric field seen by a ring by its charge, using (29):

$$F_{12} = \frac{q_1 q_2 b}{2\pi^2\varepsilon_0 r_1^3\left(1 - k^2\right)}E(k)$$
$$k = \frac{2\sqrt{Aa}}{r_1} \qquad (48)$$
$$r_1 = \sqrt{(A+a)^2 + b^2}$$

XII. EXAMPLES

Some toroids analyzed by the methods above. V_{max} was obtained from (39) and (24), except for the "holeless" toroid, where (23) and (24) were used. All the capacitances (in this and the other examples) in pF:

D x d	C_{exact}	20 rings	200 rings	V_{max} (kV)
0.2x0.1	9.6877342	9.6862459	9.6877328	226.2
0.3x0.1	13.527991	13.526517	13.527990	282.9485
0.4x0.1	17.200315	17.198812	17.200313	328.9148
0.5x0.1	20.738038	20.736480	20.738037	367.4999

Open hemispheres (D = diameter):

D	C_{exact}	20 rings	200 rings
0.2	9.1049254	9.0451871	9.0989244
0.3	13.657388	13.567781	13.648387
0.4	18.209851	18.090374	18.197849
0.5	22.762314	22.612968	22.747311

Flat disks (D = diameter):

D	C_{exact}	20 rings	200 rings
0.2	7.0833502	7.0067052	7.0757027
0.3	10.625025	10.510058	10.613554
0.4	14.166701	14.013411	14.151405
0.5	17.708376	17.516763	17.689257

Hollow cylinders (D = diameter, h = height):

D	h	20 rings	200 rings
0.2	1	27.2508153	27.5562772
0.3	1	32.7125753	33.0502066
0.4	1	37.6883716	38.0515957
0.5	1	42.3659124	42.7508210

Hollow cones (D = diameter, h = height):

D	h	20 rings	200 rings
0.2	1	20.6332474	20.8219907
0.3	1	24.3200548	24.5554254
0.4	1	27.7305014	28.0027331
0.5	1	30.9951485	31.3004301

In the last two cases no explicit formulas were found in the literature, although very probably they are known.

The general algorithm for objects with axial symmetry was implemented in the Inca program and used to generate the next examples:

A closed hemisphere can be generated by the combination of an open hemisphere and a flat disk (half of the rings for each element, D = diameter):

D	C_{exact}	20 rings	200 rings
0.2	9.4052249	9.3751321	9.4038325
0.3	14.1078374	14.0626982	14.1057488
0.4	18.8104499	18.7502642	18.8076651
0.5	23.5130623	23.4378303	23.5095813

Two spheres in contact (D = diameter, half of the rings for each sphere):

D	C_{exact}	20 rings	200 rings
0.1	7.7123025	7.7105894	7.7123007
0.2	15.4246050	15.4211788	15.4246014
0.3	23.1369075	23.1317682	23.1369021
0.4	30.8492100	30.8423576	30.8492028
0.5	38.5615125	38.5529470	38.5615035

Two different spheres in contact. Some cases are listed in [11][12]. For spheres of radii a and b, when $a = 2b$ $C = 4\pi\varepsilon_0 a \,\mathrm{Ln}\,3$, and when $a = 3b$ $C = 4\pi\varepsilon_0 a 3/4 \,\mathrm{Ln}\,4$.

a, b	Exact	40 rings	400 rings
0.1, 0.05	12.2237103	12.2233396	12.2237099
0.1, 0.0333...	11.5684538	11.5680790	11.5684534
0.1,0.025	11.3481786	11.3477839	11.3481782

The last case, $a = 4b$, is more complicated, $C = 4\pi\varepsilon_0 a\left(1/2\,\mathrm{Ln}\,5 + 2\sqrt{5}/10\,\mathrm{Ln}\left(\left(1+\sqrt{5}\right)/2\right)\right)$. The general case can be computed by the formula in [11] (in a different form) and [12]:

$$C = 4\pi\varepsilon_0 \frac{ab}{a+b}\left(-\psi\left(\frac{b}{a+b}\right) - \psi\left(\frac{a}{a+b}\right) - 2\gamma\right) \quad (49)$$

where $\psi(x)$ is the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$, derivative of the logarithm of the gamma function, and γ is Euler's constant, $\gamma \cong 0.577215665$.

A toroid with the central hole closed by a thin disk. Note the small difference to a regular toroid. A toroid where the closure of the central hole doubles the capacitance would have an aspect ratio of about 1×0.0004. Half of the rings for each element, D = major diameter, d = diameter of the tube:

D x d	20 rings	200 rings	400 rings
0.3x0.1	13.5176679	13.5296046	13.5296149
0.4x0.1	17.2225678	17.2348074	17.2348180
0.5x0.1	20.8623320	20.8748605	20.8748714

Maximum electric field between spheres with opposite voltages. Half of the rings to each sphere. Dimensions as in (33). Fields in V/m/V:

a, b, c	Exact	40 rings	400 rings
0.1, 0.1, 0.5	14.7654541	14.654655	14.762658
0.1, 0.2, 0.5	20.7165237	20.434842	20.711307
0.1, 0.3, 0.5	32.2318226	31.394734	32.219293

Capacitance matrix for two spheres. Half of the rings to each sphere. Dimensions as in (33):

k_{11} (radius a)

a, b, c	Exact	40 rings	400 rings
0.1, 0.1, 0.5	11.6112177	11.6108704	11.6112174
0.1, 0.2, 0.5	12.3051750	12.3047650	12.3051745
0.1, 0.3, 0.5	13.7605384	13.7603742	13.7605373

k_{22} (radius b)

a, b, c	Exact (pF)	40 rings	400 rings
0.1, 0.1, 0.5	11.6112177	11.6108704	11.6112174
0.1, 0.2, 0.5	24.3154312	24.3146700	24.3154303
0.1, 0.3, 0.5	38.6334041	38.6326963	38.6334025

k_{12}

a, b, c	Exact (pF)	40 rings	400 rings
0.1, 0.1, 0.5	-2.3264588	-2.3263316	-2.3264587
0.1, 0.2, 0.5	-4.9456676	-4.9454137	-4.9456673
0.1, 0.3, 0.5	-8.3626059	-8.3626805	-8.3626051

Force between the two halves of a charged sphere. The exact value is simply $F = \pi\varepsilon_0 v^2/2$, independent of the radius of the sphere. The approximate calculation is also independent of it. Values for 1 V at the sphere.

Radius (m)	Exact (pN)	40 rings	400 rings
1	13.90813	13.56727	13.87342

Force between two equal spheres at the same potential, of 1 V. The exact values were computed by the approximations in [8] and [9]. The exact value for spheres in contact is $F = 4\pi\varepsilon_0 v^2(\mathrm{Ln}\,2 - 1/4)/6$ [10].

a, b, c	Exact (pN)	40 rings	400 rings
0.1, 0.1, 0.2	8.217796	8.218122	8.217796
0.1, 0.1, 0.5	~2.996884	2.996774	2.996872
0.1, 0.1, 1.0	~0.915993	0.915949	0.915993

Force between the two halves of a horned toroid at 1 V. This case apparently has no known exact solution. The force doesn't depend on the size of the device and is proportional to the square of the potential, as happens in all cases of objects in contact. Forces in pN

D (m)	40 rings	200 rings	400 rings
1	15.55986	16.02683	16.08724

Force between two stacked horned toroids in contact, at 1V. The exact solution is unknown, but in this case the numerical analysis is expected to be precise. Forces in pN.

D (m)	40 rings	200 rings	400 rings
1	10.53158	10.53005	10.53003

The force increases as the aspect ratio of the toroids increase. It doubles for a 12.6874×1 toroid.

The problem with this approach is that as the number of rings increases it becomes more and more difficult to invert the matrix **P** with precision and in reasonable time. The examples show that the method is not very precise for objects with edges. The precision can be enhanced by adding more rings to the regions close to edges. For example, in the example of force between the two halves of a sphere, if the ±10 degrees around the equator of the sphere are modeled with 300 rings, with 100 rings for the remaining surface, the obtained force is of 13.90274 pN, with 4 correct digits. For the horned toroid, the same distribution produces 16.14268 pN, or $4\pi\varepsilon_0 \times 0.1450832$ N, possibly with similar precision.

In the last page is a table of exact toroid capacitances calculated by (9). Note that it would be enough to have a single column with normalized aspect ratios, since for a fixed aspect ratio the capacitance is directly proportional to the major (or minor) diameter.

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This document is not a published paper.

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CHANGES

9/2/2011: Corrected eq. 9 and small corrections in the text.
6/1/2012: Corrected eq. 33.
9/1/2012: Added section IX.
10/2/2012: Added eq. 37.
15/2/2012: Added section about forces.
24/2/2012: Added formula for the capacitance of two different spheres in contact.
25/2/2012: Added more examples of forces, small corrections.
7/7/2018: Better values for the capacitances of the square plate and others.

Exact toroid capacitances (diameters in meters, capacitances in pF)

Minor d. / Major d.	0.010	0.020	0.030	0.040	0.050	0.060	0.070	0.080	0.090	0.100	0.110	0.120	0.130	0.140	0.150	0.160	0.170	0.180
0.100	3.707	4.148	4.431	4.653	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------
0.125	4.468	5.001	5.340	5.598	5.816	6.010	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------
0.150	5.205	5.827	6.221	6.518	6.764	6.979	7.174	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------
0.175	5.923	6.630	7.080	7.417	7.691	7.929	8.143	8.339	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------
0.200	6.625	7.414	7.919	8.295	8.600	8.861	9.094	9.306	9.503	-------	-------	-------	-------	-------	-------	-------	-------	-------
0.225	7.313	8.182	8.740	9.156	9.492	9.778	10.029	10.258	10.469	10.667	10.854	-------	-------	-------	-------	-------	-------	-------
0.250	7.990	8.937	9.547	10.002	10.369	10.679	10.951	11.196	11.422	11.632	11.831	12.019	-------	-------	-------	-------	-------	-------
0.275	8.657	9.679	10.340	10.834	11.232	11.567	11.860	12.122	12.362	12.586	12.796	12.995	13.184	-------	-------	-------	-------	-------
0.300	9.316	10.411	11.121	11.654	12.082	12.443	12.757	13.037	13.292	13.528	13.749	13.959	14.158	14.349	-------	-------	-------	-------
0.325	9.966	11.133	11.892	12.462	12.921	13.307	13.642	13.940	14.211	14.460	14.693	14.913	15.122	15.322	15.513	15.698	-------	-------
0.350	10.609	11.846	12.653	13.260	13.749	14.160	14.517	14.833	15.119	15.383	15.628	15.858	16.077	16.285	16.485	16.678	16.863	-------
0.375	11.246	12.551	13.405	14.048	14.567	15.003	15.381	15.716	16.019	16.296	16.553	16.794	17.023	17.240	17.449	17.649	17.842	18.029
0.400	11.876	13.250	14.149	14.828	15.376	15.838	16.237	16.590	16.909	17.200	17.470	17.722	17.961	18.187	18.404	18.612	18.812	19.006
0.425	12.502	13.941	14.886	15.600	16.177	16.663	17.084	17.456	17.791	18.096	18.379	18.643	18.891	19.126	19.351	19.567	19.775	19.976
0.450	13.122	14.627	15.616	16.365	16.970	17.481	17.922	18.313	18.664	18.984	19.280	19.555	19.814	20.059	20.292	20.516	20.731	20.938
0.475	13.737	15.306	16.340	17.122	17.756	18.291	18.753	19.162	19.530	19.865	20.173	20.460	20.729	20.984	21.226	21.457	21.680	21.894
0.500	14.348	15.981	17.057	17.874	18.535	19.093	19.577	20.005	20.389	20.738	21.060	21.358	21.638	21.902	22.153	22.392	22.622	22.844
0.525	14.955	16.650	17.769	18.619	19.308	19.890	20.394	20.840	21.240	21.604	21.939	22.250	22.540	22.814	23.074	23.322	23.559	23.787
0.550	15.558	17.315	18.476	19.358	20.075	20.680	21.204	21.668	22.085	22.464	22.812	23.135	23.436	23.720	23.989	24.245	24.489	24.725
0.575	16.157	17.975	19.177	20.092	20.836	21.464	22.009	22.491	22.924	23.317	23.678	24.013	24.326	24.619	24.897	25.162	25.414	25.657
0.600	16.753	18.631	19.874	20.821	21.591	22.242	22.807	23.307	23.756	24.164	24.539	24.886	25.209	25.513	25.800	26.073	26.334	26.584
0.625	17.346	19.284	20.567	21.546	22.342	23.016	23.601	24.118	24.583	25.006	25.393	25.752	26.087	26.401	26.698	26.979	27.248	27.505
0.650	17.935	19.932	21.256	22.265	23.088	23.784	24.388	24.924	25.405	25.842	26.243	26.614	26.959	27.284	27.590	27.880	28.157	28.421
0.675	18.522	20.577	21.940	22.981	23.829	24.547	25.171	25.724	26.221	26.672	27.086	27.469	27.826	28.161	28.477	28.776	29.060	29.333
0.700	19.105	21.218	22.621	23.692	24.566	25.306	25.949	26.520	27.032	27.498	27.925	28.320	28.688	29.033	29.358	29.666	29.959	30.239
0.725	19.686	21.856	23.298	24.399	25.298	26.060	26.723	27.310	27.838	28.318	28.759	29.166	29.545	29.900	30.235	30.552	30.853	31.141
0.750	20.265	22.492	23.971	25.103	26.027	26.810	27.492	28.097	28.640	29.134	29.588	30.007	30.397	30.763	31.107	31.433	31.742	32.037
0.775	20.841	23.124	24.641	25.803	26.751	27.556	28.257	28.879	29.438	29.946	30.412	30.843	31.245	31.621	31.974	32.309	32.627	32.930
0.800	21.414	23.753	25.308	26.499	27.472	28.299	29.018	29.657	30.231	30.753	31.232	31.675	32.088	32.474	32.837	33.181	33.507	33.818
0.825	21.985	24.379	25.972	27.192	28.190	29.037	29.775	30.430	31.020	31.556	32.048	32.503	32.926	33.323	33.696	34.048	34.383	34.702
0.850	22.554	25.003	26.633	27.882	28.904	29.772	30.529	31.200	31.805	32.355	32.859	33.326	33.761	34.167	34.550	34.912	35.255	35.581
0.875	23.121	25.625	27.292	28.569	29.615	30.504	31.279	31.967	32.586	33.150	33.667	34.146	34.591	35.008	35.400	35.771	36.122	36.457
0.900	23.686	26.244	27.947	29.253	30.323	31.232	32.025	32.729	33.364	33.941	34.471	34.961	35.417	35.845	36.246	36.626	36.986	37.328
0.925	24.249	26.860	28.600	29.934	31.027	31.957	32.768	33.489	34.138	34.728	35.271	35.773	36.240	36.677	37.089	37.477	37.846	38.196
0.950	24.810	27.474	29.250	30.613	31.729	32.679	33.508	34.245	34.908	35.513	36.068	36.581	37.059	37.507	37.927	38.325	38.702	39.060
0.975	25.369	28.086	29.898	31.288	32.428	33.398	34.245	34.997	35.676	36.293	36.861	37.386	37.874	38.332	38.762	39.169	39.554	39.920
1.000	25.927	28.696	30.543	31.962	33.124	34.114	34.979	35.747	36.440	37.071	37.650	38.187	38.686	39.154	39.594	40.009	40.403	40.777
1.025	26.482	29.304	31.186	32.632	33.818	34.827	35.709	36.494	37.201	37.845	38.437	38.985	39.495	39.973	40.422	40.846	41.248	41.630
1.050	27.036	29.910	31.827	33.300	34.509	35.538	36.437	37.237	37.959	38.616	39.220	39.779	40.300	40.788	41.246	41.680	42.090	42.480
1.075	27.588	30.514	32.466	33.966	35.197	36.246	37.163	37.978	38.714	39.384	40.000	40.571	41.102	41.600	42.068	42.510	42.929	43.327
1.100	28.139	31.116	33.103	34.630	35.883	36.951	37.885	38.716	39.466	40.149	40.778	41.360	41.901	42.409	42.886	43.337	43.764	44.170
1.125	28.688	31.716	33.737	35.291	36.567	37.654	38.605	39.452	40.215	40.912	41.552	42.145	42.698	43.215	43.701	44.161	44.596	45.010
1.150	29.236	32.315	34.370	35.950	37.248	38.355	39.323	40.185	40.962	41.671	42.324	42.928	43.491	44.018	44.513	44.982	45.425	45.847
1.175	29.782	32.911	35.001	36.608	37.927	39.053	40.038	40.915	41.706	42.428	43.092	43.708	44.281	44.818	45.323	45.800	46.252	46.681
1.200	30.327	33.506	35.629	37.263	38.604	39.749	40.751	41.643	42.448	43.183	43.859	44.485	45.068	45.615	46.129	46.615	47.075	47.513
1.225	30.871	34.100	36.256	37.916	39.279	40.443	41.461	42.368	43.187	43.935	44.622	45.259	45.853	46.409	46.933	47.427	47.895	48.341
1.250	31.413	34.692	36.882	38.567	39.952	41.134	42.169	43.091	43.924	44.684	45.383	46.031	46.635	47.201	47.734	48.237	48.713	49.166
1.275	31.953	35.282	37.505	39.216	40.623	41.824	42.875	43.812	44.658	45.431	46.142	46.801	47.415	47.990	48.532	49.043	49.528	49.989
1.300	32.493	35.871	38.127	39.864	41.292	42.511	43.579	44.531	45.390	46.175	46.898	47.568	48.192	48.777	49.327	49.848	50.341	50.809
1.325	33.031	36.458	38.747	40.510	41.959	43.197	44.281	45.247	46.120	46.918	47.652	48.332	48.966	49.561	50.121	50.649	51.150	51.627
1.350	33.568	37.044	39.365	41.154	42.624	43.880	44.980	45.961	46.848	47.658	48.403	49.094	49.739	50.343	50.911	51.448	51.957	52.442
1.375	34.104	37.628	39.982	41.796	43.287	44.562	45.678	46.674	47.574	48.395	49.152	49.854	50.509	51.122	51.699	52.245	52.762	53.254
1.400	34.639	38.211	40.598	42.436	43.949	45.241	46.374	47.384	48.297	49.131	49.899	50.612	51.276	51.899	52.485	53.039	53.565	54.064
1.425	35.172	38.792	41.212	43.075	44.609	45.919	47.068	48.092	49.019	49.865	50.644	51.367	52.041	52.674	53.269	53.831	54.364	54.872
1.450	35.705	39.373	41.824	43.713	45.267	46.595	47.760	48.799	49.738	50.596	51.387	52.120	52.805	53.446	54.050	54.621	55.162	55.677
1.475	36.236	39.952	42.435	44.349	45.923	47.270	48.450	49.503	50.456	51.326	52.128	52.872	53.566	54.216	54.829	55.408	55.957	56.480
1.500	36.766	40.530	43.045	44.983	46.578	47.942	49.138	50.206	51.171	52.053	52.866	53.621	54.324	54.984	55.606	56.193	56.751	57.280