

Capacitance Calculations

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Abstract—This document describes calculation methods for distributed capacitances of objects with several particular shapes, and methods for the evaluation of the electric field around them. It's fundamentally a collection of formulas, some not very easy to find in the literature. The algorithms were implemented in the Inca program, available at <http://www.coe.ufrj.br/~acmq/programs>.

I. INTRODUCTION

Most of the formulas below are known since long time, most dating from works in the XIX century. Some appear in Maxwell's book [1], and some in other collections of explicit formulas for electromagnetic problems, as [2], or in other early works as [3]-[5]. In most cases I have just adapted the notation, but some derivations not found in other works are presented too.

In most of the early works, capacitance is expressed in units of length. For example, the capacitance of a sphere of radius a in free space is listed in [1] and [2] as $C=a$. To convert this unit to Farads, it's necessary to multiply the value by $4\pi\epsilon_0$, where ϵ_0 is the permittivity of vacuum, $\epsilon_0 = 8.8541878 \times 10^{-12}$. ϵ_0 can be calculated from the speed of light c and from the magnetic permeability of vacuum, $\mu_0 = 4\pi \times 10^{-7}$ (a definition), from the relation:

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \quad (1)$$

The capacitance of a sphere of radius a meters is then:

$$C_{\text{sphere}} = 4\pi\epsilon_0 a = 111.26501a \text{ pF} \quad (2)$$

Other figures that have simple expressions for the free-space capacitance are:

A thin flat disk with radius a [2]:

$$C_{\text{disk}} = 8\epsilon_0 a = 70.833503a \text{ pF} \quad (3)$$

An open hemisphere with radius a [2]:

$$C_{\text{open hemisphere}} = 4\pi\epsilon_0 a(1/2+1/\pi) = 91.049254a \text{ pF} \quad (4)$$

A closed (with a flat disk) hemisphere with radius a [2]:

$$C_{\text{closed hemisphere}} = 8\pi\epsilon_0 a(1-1/\sqrt{3}) = 94.052249 \text{ pF} \quad (5)$$

Two spheres with radius a in contact [1]:

$$C_{\text{two spheres}} = 8\pi\epsilon_0 a \text{Ln}(2) = 154.24505a \text{ pF} \quad (6)$$

An "oblate spheroid" is the figure generated by the rotation of an ellipse around its minor axis. A "prolate spheroid" is generated by the rotation of an ellipse around its major axis. The capacitances of these figures are, considering the major axis with length $2a$ and the minor axis with the length $2b$ [2]:

$$C_{\text{oblate}} = 4\pi\epsilon_0 \frac{\sqrt{a^2 - b^2}}{\sin^{-1} \frac{\sqrt{a^2 - b^2}}{a}} \quad (7)$$

$$C_{\text{prolate}} = 4\pi\epsilon_0 \frac{\sqrt{a^2 - b^2}}{\ln \frac{a + \sqrt{a^2 - b^2}}{b}} \quad (8)$$

Note the limits when $a = b$ reducing to (2), and the reduction to (3) when $b = 0$ in (7).

For bodies embedded in materials with other permittivities, it's just a question of multiplying ϵ_0 by the relative permittivity ϵ of the material. The case when different dielectrics are present on the structure will be not discussed here.

II. CAPACITANCE OF A TOROID

From [2] (the same formula appears in [3], that is probably the origin of this formula, but in a somewhat different notation) the capacitance of a toroid with major diameter D and minor diameter d , $d < D/2$, (fig. 1) is:

$$C = 16\epsilon_0 \sqrt{A^2 - a^2} \sum_{n=0}^{\infty} \sigma_n \frac{Q_{n-1/2}(x)}{P_{n-1/2}(x)}, \quad (9)$$

$$\sigma_n = 1/2 \text{ for } n = 0, 1 \text{ for } n > 0;$$

$$A = \frac{D-d}{2}; \quad a = \frac{d}{2}; \quad x = \frac{A}{a}$$

where $P_{n-1/2}(x)$ and $Q_{n-1/2}(x)$ are Legendre functions, or in this case, "toroidal functions".

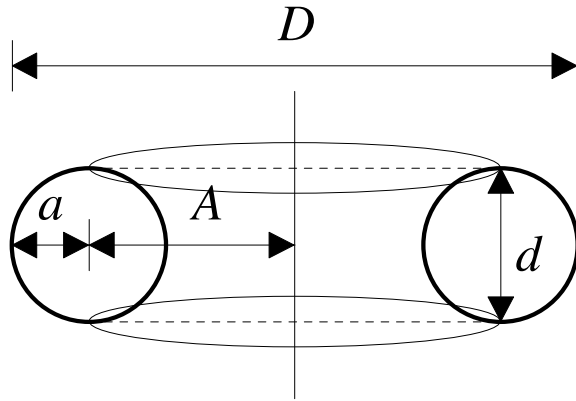


Fig. 1. Toroid with major diameter D , minor diameter d , center radius A , and tube radius a .

These functions can be evaluated in the following way: The first two terms can be obtained from their relations with the complete elliptic integrals of first and second kinds:

$$\begin{aligned} Q_{-1/2}(x) &= kK; \\ Q_{1/2}(x) &= 2\frac{K-E}{k} - kK; \\ P_{-1/2}(x) &= \frac{2}{\pi} kK'; \\ P_{1/2}(x) &= \frac{2}{\pi} \left(\frac{2E'}{k} - kK' \right) \end{aligned} \quad (10)$$

The modulus for the elliptic integrals K and E is:

$$k = \sqrt{\frac{2a}{A+a}} \quad (11)$$

And for the elliptic integrals K' and E' (evaluated in the same way, with modulus k'):

$$k' = \sqrt{1-k^2} \quad (12)$$

This is enough for the evaluation of the two first terms of the series (enough for thin toroids). The other terms can be obtained using the recursion for Legendre functions, identical for both functions:

$$\begin{aligned} P_{m+1/2}(x) &= \frac{2mxP_{m-1/2}(x) - (m-1/2)P_{m-3/2}(x)}{m+1/2}; \\ Q_{m+1/2}(x) &= \frac{2mxQ_{m-1/2}(x) - (m-1/2)Q_{m-3/2}(x)}{m+1/2} \end{aligned} \quad (13)$$

where $m=n-1$. All the terms can then be easily computed, starting with $n=2$ in the series (7), or $m=1$.

The complete elliptic integrals are the irreducible functions:

$$\begin{aligned} K &= F(k, \pi/2) = F(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} \\ E &= E(k, \pi/2) = E(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \varphi} d\varphi \end{aligned} \quad (14)$$

They can be quickly and precisely evaluated using the arithmetic-geometric mean method, below implemented in a Pascal routine:

```
{
Complete elliptic integrals of first
and second classes - AGM method.
Returns the global variables:
Ek=E(c) and Fk=F(c) (E and K)
Doesn't require more than 7 iterations for
c between 0 and 0.9999999999.
Reference: Pi and the AGM, J. Borwein and
P. Borwein, John Wiley & Sons.
}
procedure EF(c:real);
var
  a,b,a1,b1,E,i:real;
begin
  a:=1;
  b:=sqrt(1-sqr(c));
  E:=1-sqr(c)/2;
  i:=1;
  repeat
    a1:=(a+b)/2;
    b1:=sqrt(a*b);
    E:=E-i*sqr((a-b)/2);
    i:=2*i;
    a:=a1;
    b:=b1;
  until abs(a-b)<1e-15;
  Fk:=pi/(2*a);
  Ek:=E*Fk;
end;
```

III. APPROXIMATE CALCULATIONS FOR PARTIAL TOROIDS

A partial toroid can be described as a surface generated by the revolution of a partial circle of radius a centered at a distance A along the radial axis r from the revolution axis z . The circle limits are defined by two angles θ_1 and θ_2 . See fig. 2.

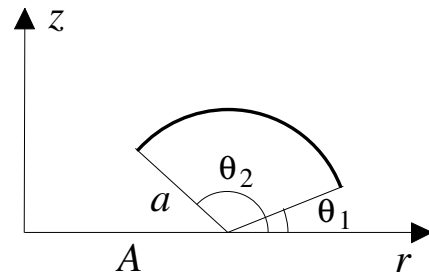


Fig. 2: A partial toroidal surface is generated by the rotation of a partial circle around the vertical axis.

With this formulation several figures can be generated, as a regular toroid when $\theta_2 - \theta_1 = 2\pi$ and $a < A$, a sphere when $A = 0$, $\theta_2 = -\theta_1 = \pi/2$, an open hemisphere, etc. Even overlapping toroids, with $a > A$, can be generated.

This surface can be decomposed in a set of n infinitely thin circles with axes at the z axis, positioned at heights z_i , and with radii r_i , uniformly spaced at angles $\Delta\theta$ along the surface:

$$\begin{aligned} \Delta\theta &= \frac{\theta_2 - \theta_1}{n} \\ \theta_i &= \theta_1 + \frac{\Delta\theta}{2} + (i-1)\Delta\theta, \quad i=1\dots n \\ r_i &= A + a \cos \theta_i \\ z_i &= a \sin \theta_i \end{aligned} \quad (15)$$

Each of these rings has a uniform charge distribution, with a total charge q_i . The potential Ψ due to each ring i at any given position r_0, z_0 is given by:

$$\begin{aligned} \Psi_i(r_0, z_0) &= \frac{q_i}{4\pi^2 \epsilon_0 \sqrt{|r_0 r_i|}} Q_{-1/2} \left(\frac{2}{k^2} - 1 \right) = \frac{q_i}{2\pi^2 \epsilon_0 R_1} K \\ k &= \frac{2\sqrt{|r_0 r_i|}}{R_1} \\ R_1 &= \sqrt{(|r_i| + |r_0|)^2 + (z_0 - z_i)^2} \end{aligned} \quad (16)$$

The absolute values allow correct treatment of the cases when some radii are negative.

Considering then the mutual influences among all the rings, a matrix \mathbf{P} can be computed, that allows the calculation of the potentials v_i at each ring, once the charges q_i are known [1]:

$$\begin{aligned} \mathbf{v} &= \mathbf{P}\mathbf{q} \\ P_{ij} &= P_{ji} = \Psi_i(r_j, z_j) / q_j \\ P_{ii} &= \Psi_i(r_i, z_i + R) / q_i \end{aligned} \quad (17)$$

For the calculation of the "self-potentials" P_{ii} , something must be assumed about the radius of the rings, R . The formulation calculates then the potentials at a distance R above the rings. The maximum physically possible value of R would be when adjacent rings touch:

$$R_{\max} = a \sin \frac{\Delta\theta}{2} \quad (18)$$

Any reasonable fraction of this value can be used with similar results, but there is one that produces better results in the next calculation, that was found (by trying!) to, curiously, be:

$$R = \frac{a}{\pi} \sin \frac{\Delta\theta}{2} \quad (19)$$

This radius makes the area of the surface of the ring to be identical to the flat area represented by it, at least in the cases when $\theta = n\pi/2$ and small $\Delta\theta$ (as in the equator and poles of a sphere split in many rings).

The charge distribution for uniform potential V at all the rings can be calculated by inverting the matrix \mathbf{P} . The total charge in each ring is then obtained from a sum of the corresponding lines of the inverse of \mathbf{P} , \mathbf{C} . The coefficients of \mathbf{C} are the influence coefficients k_{ij} :

$$\begin{aligned} q_i &= V \sum_{j=1}^n k_{ij} \\ \mathbf{C} &= \mathbf{P}^{-1} \end{aligned} \quad (20)$$

And the capacitance of the whole assembly is simply the sum of all the elements of \mathbf{C} :

$$C_{\text{total}} = \sum_{i=1}^n \sum_{j=1}^n k_{ij} \quad (21)$$

Surface electric field

The electric field at any point of the surface is normal to it and can be calculated by Gauss' law as proportional to the charge density at that point of the surface:

$$E_i = \frac{\rho_i}{\epsilon_0} \quad (22)$$

where ρ_i is the surface charge density, uniform around the ring i . For a closed surface, the electric field is entirely at the outer surface. In this case, it can be calculated directly from the charge distribution alone.

Assuming constant voltage at the surface of the object, the charges at the rings can be calculated by (20). The ring i has a length $2\pi r_i$ and a total charge q_i . The ring represents a thin belt with width equal to $a\Delta\theta$. The charge density and the electric field in a small length l are then:

$$\begin{aligned} \rho_i &= \frac{q_i}{2\pi r_i} \frac{1}{la\Delta\theta} = \frac{q_i}{2\pi r_i a \Delta\theta} \\ E_i &= \frac{q_i}{2\pi r_i a \Delta\theta \epsilon_0} \end{aligned} \quad (23)$$

An important application of this calculation is the determination of the breakout voltage of the object, the voltage that causes ionization of the air around it when the electric field reaches about 3 MV/m:

$$V_{\max} = 3000 / \text{Max } E_i \text{ kV} \quad (24)$$

For a toroid, this value occurs at the maximum diameter. In the case of open objects, it's not possible to calculate the surface electric field in this way, because it is split in an unknown way between the two sides of the surface. The calculation is also meaningless if the object

has an edge, where the electric field is ideally infinite. A strange problem with (23) is that it fails when the rings are close to the center of a spherical surface. The last ring appears to have significantly less charge than it should have (around 92%). The calculations for capacitance, however, continue to result in good values.

IV. GENERAL TRUNCATED CONES

Any other figure with circular symmetry can be analyzed by the same method. A simple case is the revolution of a straight line around the central axle, that generates figures ranging from a flat disk with a possible central hole to a cone or an open cylinder.

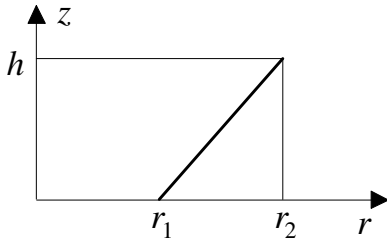


Fig. 3. A line that rotates around the vertical axis.

The coordinates of the rings are then:

$$\begin{aligned}\Delta z &= \frac{h}{n} \\ \Delta r &= \frac{r_2 - r_1}{n} \\ z_i &= \frac{\Delta z}{2} + (i-1)\Delta z, \quad i = 1 \dots n \\ r_i &= r_1 + \frac{\Delta r}{2} + (n-1)\Delta r, \quad i = 1 \dots n\end{aligned}\quad (25)$$

The radius to use in the calculation of the self-potentials would be, still using the maximum divided by π :

$$R = \frac{\sqrt{(r_2 - r_1)^2 + h^2}}{2n\pi} \quad (26)$$

With this radius, the surface charge density and the surface electric field (for a closed object) can be calculated considering that the surface area of the ring is identical to the belt area represented by it, what is approximately valid also for the case of curves, using R given by (19):

$$\begin{aligned}\rho_i &= \frac{q_i}{2\pi r_i} \frac{1}{l} = \frac{q_i}{4\pi^2 r_i R} \\ E_i &= \frac{q_i}{4\pi^2 r_i R \epsilon_0}\end{aligned}\quad (27)$$

V. ELECTRIC FIELD FROM A RING

The electric field anywhere can be calculated by adding the electric fields due to the rings. From (16), the radial and axial components of the electric field can be calculated by differentiation, resulting in:

$$E_{radial} = -\frac{d\Psi_i}{dr_0} = -\frac{q_i \text{sign } r_i}{2\pi^2 \epsilon_0 R_1^3} \left(-(|r_i| - |r_0|)K + \frac{E - k'^2 K}{kk'^2} \left(\frac{2|r_0|}{k} - k(|r_i| - |r_0|) \right) \right) \quad (28)$$

$$E_{axial} = -\frac{d\Psi_i}{dz_0} = \frac{q_i(z_i - z_0)E}{2\pi^2 \epsilon_0 R_1^3 k'^2} \quad (29)$$

$$E_{total} = \sqrt{E_{radial}^2 + E_{axial}^2} \quad (30)$$

where the derivative of the elliptic integral K in relation to the modulus k was used (the derivative of E is listed below too for reference, but was not necessary):

$$\begin{aligned}\frac{dK}{dk} &= \frac{E - k'^2 K}{kk'^2}; \quad \frac{dE}{dk} = \frac{E - K}{k} \\ k'^2 &= 1 - k^2\end{aligned}\quad (31)$$

VI. GENERAL CASE WITH AXIAL SYMMETRY

The capacitance matrix and the potential and electric field around a series of objects with axial symmetry decomposed in thin rings can then be easily calculated. The objects are decomposed in series of partial toroids conical sheets, and other shapes (as ellipses) and these parts are decomposed in rings. To obtain the capacitance matrix, it's just a question of adding the terms of the total capacitance matrix that correspond to the rings that belong to the objects, instead of adding them all to obtain the capacitance of the entire object. The charges in all the rings can be obtained from the complete equation $\mathbf{q} = \mathbf{C}\mathbf{V}$, with the assigned voltages in the objects arranged in \mathbf{V} in correspondence with the rings that belong to the objects. The potential anywhere around the objects is obtained by adding (16) for all the rings, and the electric field by adding (28) and (29) and using (30). The terms at the diagonal of the capacitance matrix correspond to the capacitances of the objects to ground when all the other objects are grounded too. The influence coefficients out of the diagonal measure the relation between the charge induced in one object and the voltage in another, when all the other objects are grounded. From the capacitance matrix, a model of the circuit using lumped capacitors can be derived, by observing the equivalence:

$$\mathbf{C} = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{12} & k_{22} & \dots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1n} & k_{2n} & \dots & k_{nn} \end{bmatrix} = \begin{bmatrix} C_1 + C_{12} + \dots + C_{1n} & -C_{12} & \dots & -C_{1n} \\ -C_{12} & C_2 + C_{12} + \dots + C_{2n} & \dots & -C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -C_{1n} & -C_{2n} & \dots & C_n + C_{n-1} + \dots + C_{1n} \end{bmatrix} \quad (32)$$

C_1, C_2, \dots, C_n are direct capacitances between the elements and the ground, and the other elements are the negative of the floating capacitances between the objects. The direct capacitance to ground for the object i is just the sum of the elements in the line, or column, i of \mathbf{C} .

VII. MAXIMUM ELECTRIC FIELD BETWEEN TWO SPHERES

A good test for these field calculations is the known formula for the maximum electric field between two different spheres [4]. The expression comes directly from the method of images developed by Lord Kelvin. For two spheres of radii a and b , $a < b$, with distance between centers c , at potentials v_1 and v_2 , the maximum electric field at the surface of the smaller sphere (assumed as being where the surface of the smaller sphere intercepts the line between the centers of the spheres) is given by:

$$E_{\max} = \frac{(1+\xi)^2}{a(1-\xi)} \left[v_1 \left\{ \frac{1-\xi}{(1+\xi)^2} + \alpha \frac{1-\xi\alpha^2}{(1+\xi\alpha^2)^2} + \alpha^2 \frac{1-\xi\alpha^4}{(1+\xi\alpha^4)^2} + \dots \right\} - v_2 \left\{ \eta \frac{1-\eta\alpha}{(1+\eta\alpha)^2} + \alpha\eta \frac{1-\eta\alpha^3}{(1+\eta\alpha^3)^2} + \alpha^2\eta \frac{1-\eta\alpha^5}{(1+\eta\alpha^5)^2} + \dots \right\} \right] \quad (33)$$

$$\xi = \frac{a+b\alpha}{c}; \quad \eta = \frac{b+a\alpha}{c};$$

$$\alpha = \text{root} < 1 \text{ of } (a\alpha + b)(b\alpha + a) = c^2\alpha$$

This formula converges slowly when the spheres are at small distance, but the speed is acceptable. [4] develops a better expression for the case of spheres at small distance too.

VIII. CAPACITANCES OF TWO SPHERES

Similar formulas, due to Kirchhoff, lead to the capacitance matrix of two spheres [5]. For two spheres with radii a and b and distance between centers c :

$$k_{11} = 8\pi\epsilon_0\lambda \left[\frac{\xi}{1+\xi^2} + \frac{\alpha\xi}{1+\alpha^2\xi^2} + \frac{\alpha^2\xi}{1+\alpha^4\xi^2} + \dots \right];$$

$$k_{22} = 8\pi\epsilon_0\lambda \left[\frac{\eta}{1+\eta^2} + \frac{\alpha\eta}{1+\alpha^2\eta^2} + \frac{\alpha^2\eta}{1+\alpha^4\eta^2} + \dots \right]; \quad (34)$$

$$k_{12} = -8\pi\epsilon_0\lambda \left[\frac{\alpha}{1+\alpha^2} + \frac{\alpha^2}{1+\alpha^4} + \frac{\alpha^3}{1+\alpha^6} + \dots \right];$$

$$\xi = \sqrt{1 + \frac{\lambda^2}{a^2}} - \frac{\lambda}{a}; \quad \alpha = \frac{c\xi - a}{b}; \quad \eta = \frac{\alpha}{\xi};$$

$$\lambda = \frac{\sqrt{(c+a+b)(c-a-b)(c+a-b)(c-a+b)}}{2c}$$

These formulas also converge slowly when the spheres are at small distance. [5] shows a better formula for small distances.

The coefficients of the capacitance matrix represent the ratio between the induced charges and the voltages. k_{11} and k_{12} represent capacitances to ground from a sphere with the other sphere grounded, and $-k_{12}$ is the floating capacitance between the spheres. The differential capacitance between the spheres is obtained by assuming opposite charges $\pm q$ on them:

$$C_{\text{diff}} = \frac{v_1 - v_2}{q} = \frac{k_{11}k_{22} - k_{12}^2}{k_{11} + k_{22} + 2k_{12}} \quad (35)$$

The capacitances to ground with the other sphere floating can also be calculated, by assuming zero charge in the floating sphere:

$$C_{10} = \frac{k_{11}k_{22} - k_{12}^2}{k_{22}}; \quad C_{20} = \frac{k_{11}k_{22} - k_{12}^2}{k_{11}} \quad (36)$$

IX. POTENTIAL AND ELECTRIC FIELD AROUND A TOROID

The solution of this problem can be traced to [3]. The formula for the potential also appears in [2]. The potential around an isolated toroid in free space, with central radius A and tube radius a , at a radial distance r and axial distance z from the center, is found as:

$$\Psi(\alpha, \beta) = \frac{2V}{\pi} \sqrt{2(\cosh \beta - \cos \alpha)} \sum_{n=0}^{\infty} \sigma_n \frac{Q_{n-1/2}(x)}{P_{n-1/2}(x)} P_{n-1/2}(\cosh \beta) \cos n\alpha;$$

$$\sigma_n = 1/2 \text{ for } n=0, 1 \text{ for } n > 0;$$

$$x = \frac{A}{a}; \quad c = \sqrt{A^2 - a^2};$$

$$\beta = \frac{1}{2} \ln \frac{z^2 + (r+c)^2}{z^2 + (r-c)^2};$$

$$\sin \alpha = \frac{z}{r} \sinh \beta; \quad \cos \alpha = \cosh \beta - \frac{c}{r} \sinh \beta$$

$$\alpha = \tan^{-1} \frac{\sin \alpha}{\cos \alpha} = \tan^{-1} \frac{2cz}{r^2 + z^2 - c^2}; \quad (37)$$

The surface electric field can be found by the differentiation of (37). The maximum occurs when $\cosh \beta = x = A/a$ (toroid surface) and $\alpha=0$ (major diameter). The result, hinted in [3] but not developed, is the series:

$$E = \frac{4\sqrt{2}V(x - \cos \alpha)^{3/2}}{\pi d(x^2 - 1)} \sum_{n=0}^{\infty} \sigma_n \frac{\cos n\alpha}{P_{n-1/2}(x)} \quad (38)$$

The ideal exact breakdown voltage can then be obtained as in (24). This series converges somewhat more slowly than (9) but still can achieve high precision. The series (37) may lose precision due to errors in the evaluation of $Q_{n+1/2}(x)$ by the recursion (13).

X. EXAMPLES

Some toroids analyzed by the methods above. V_{\max} was obtained from (38) and (24), except for the “holeless” toroid, where (23) and (24) were used. All the capacitances (in this and the other examples) in pF:

$D \times d$	C_{exact}	20 rings	200 rings	V_{\max} (kV)
0.2x0.1	9.6877342	9.6862459	9.6877328	226.2
0.3x0.1	13.527991	13.526517	13.527990	282.9485
0.4x0.1	17.200315	17.198812	17.200313	328.9148
0.5x0.1	20.738038	20.736480	20.738037	367.4999

Open hemispheres ($D = \text{diameter}$):

D	C_{exact}	20 rings	200 rings
0.2	9.1049254	9.0451871	9.0989244
0.3	13.657388	13.567781	13.648387
0.4	18.209851	18.090374	18.197849
0.5	22.762314	22.612968	22.747311

Flat disks ($D = \text{diameter}$):

D	C_{exact}	20 rings	200 rings
0.2	7.0833502	7.0067052	7.0757027
0.3	10.625025	10.510058	10.613554
0.4	14.166701	14.013411	14.151405
0.5	17.708376	17.516763	17.689257

Hollow cylinders ($D = \text{diameter}$, $h = \text{height}$):

D	h	20 rings	200 rings
0.2	1	27.2508153	27.5562772
0.3	1	32.7125753	33.0502066
0.4	1	37.6883716	38.0515957
0.5	1	42.3659124	42.7508210

Hollow cones ($D = \text{diameter}$, $h = \text{height}$):

D	h	20 rings	200 rings
0.2	1	20.6332474	20.8219907
0.3	1	45.0755428	24.5554255
0.4	1	27.7305014	28.0027331
0.5	1	30.9951485	31.3004301

In the last two cases no explicit formulas were found in the literature, although very probably they are known.

The general algorithm for objects with axial symmetry was implemented in the Inca program and used to generate the next examples:

A closed hemisphere can be generated by the combination of an open hemisphere and a flat disk (half of the rings for each element, $D = \text{diameter}$):

D	C_{exact}	20 rings	200 rings
0.2	9.4052249	9.3751321	9.4038325
0.3	14.1078374	14.0626982	14.1057488
0.4	18.8104499	18.7502642	18.8076651
0.5	23.5130623	23.4378303	23.5095813

Two spheres in contact ($D = \text{diameter}$, half of the rings for each sphere):

D	C_{exact}	20 rings	200 rings
0.1	7.7123025	7.7105894	7.7123007
0.2	15.4246050	15.4211788	15.4246014
0.3	23.1369075	23.1317682	23.1369021
0.4	30.8492100	30.8423576	30.8492028
0.5	38.5615125	38.5529470	38.5615035

A toroid with the central hole closed by a thin disk. Note the small difference to a regular toroid. A toroid where the closure of the central hole doubles the capacitance would have an aspect ratio of about 1×0.0004 . Half of the rings for each element, $D = \text{major diameter}$, $d = \text{diameter of the tube}$:

$D \times d$	20 rings	200 rings	400 rings
0.3x0.1	13.5176679	13.5296046	13.5296149
0.4x0.1	17.2225678	17.2348074	17.2348180
0.5x0.1	20.8623320	20.8748605	20.8748714

Maximum electric field between spheres with opposite voltages. Half of the rings to each sphere. Dimensions as in (33). Fields in V/m/V:

a, b, c	Exact	40 rings	400 rings
0.1, 0.1, 0.5	14.7654541	14.654655	14.762658
0.1, 0.2, 0.5	20.7165237	20.434842	20.711307
0.1, 0.3, 0.5	32.2318226	31.394734	32.219293

Capacitance matrix for two spheres. Half of the rings to each sphere. Dimensions as in (33):

k_{11} (radius a)			
a, b, c	Exact	40 rings	400 rings
0.1, 0.1, 0.5	11.6112177	11.6108704	11.6112174
0.1, 0.2, 0.5	12.3051750	12.3047650	12.3051745
0.1, 0.3, 0.5	13.7605384	13.7603742	13.7605373
k_{22} (radius b)			
a, b, c	Exact (pF)	40 rings	400 rings
0.1, 0.1, 0.5	11.6112177	11.6108704	11.6112174
0.1, 0.2, 0.5	24.3154312	24.3146700	24.3154303
0.1, 0.3, 0.5	38.6334041	38.6326963	38.6334025

k_{12}			
a, b, c	Exact (pF)	40 rings	400 rings
0.1, 0.1, 0.5	-2.3264588	-2.3263316	-2.3264587
0.1, 0.2, 0.5	-4.9456676	-4.9454137	-4.9456673
0.1, 0.3, 0.5	-8.3626059	-8.3626805	-8.3626051

The problem with this approach is that as the number of rings increases it becomes more and more difficult to invert the matrix \mathbf{P} with precision and in reasonable time.

In the next page is a table of exact toroid capacitances calculated by (9). Note that it would be enough to have a single column with normalized aspect ratios, since for a fixed aspect ratio the capacitance is directly proportional to the major (or minor) diameter.

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This document is not a published paper.

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CHANGES

9/2/2011: Corrected eq. 9 and small corrections in the text.

6/1/2012: Corrected eq. 33.

Exact toroid capacitances (diameters in meters, capacitances in pF)

Minor d. Major d.	0.010	0.020	0.030	0.040	0.050	0.060	0.070	0.080	0.090	0.100	0.110	0.120	0.130	0.140	0.150	0.160	0.170	0.180	
0.100	3.707	4.148	4.431	4.653	4.816	6.010	7.174	8.339	9.503	10.667	10.854	11.831	12.019	13.184	14.349	15.513	16.678	17.842	19.006
0.125	4.468	5.001	5.340	5.598	5.816	6.010	7.174	8.339	9.503	10.667	10.854	11.831	12.019	13.184	14.349	15.513	16.678	17.842	19.006
0.150	5.205	5.827	6.221	6.518	6.764	6.979	8.143	9.308	10.473	11.638	11.831	12.808	13.001	14.166	15.331	16.496	17.661	18.826	19.991
0.175	5.923	6.630	7.080	7.417	7.691	7.929	9.094	10.259	11.424	12.589	12.796	13.773	14.001	15.166	16.331	17.496	18.661	19.826	20.991
0.200	6.625	7.414	7.919	8.295	8.600	8.861	10.026	11.191	12.356	13.521	13.738	14.715	14.953	16.118	17.283	18.448	19.613	20.778	21.943
0.225	7.313	8.182	8.740	9.156	9.492	9.778	10.943	12.108	13.273	14.438	14.666	15.643	15.891	17.056	18.221	19.386	20.551	21.716	22.881
0.250	7.990	8.937	9.547	10.002	10.369	10.679	11.844	13.009	14.174	15.339	15.577	16.554	16.812	18.000	19.188	20.376	21.564	22.752	23.940
0.275	8.657	9.679	10.340	10.834	11.232	11.567	12.732	13.907	15.082	16.257	16.505	17.482	17.750	19.000	20.250	21.500	22.750	24.000	25.250
0.300	9.316	10.411	11.121	11.654	12.082	12.443	13.608	14.783	15.958	17.133	17.391	18.368	18.646	20.000	21.250	22.500	23.750	25.000	26.250
0.325	9.966	11.133	11.892	12.462	12.921	13.307	14.472	15.647	16.822	18.007	18.275	19.252	19.540	21.000	22.250	23.500	24.750	26.000	27.250
0.350	10.609	11.846	12.653	13.260	13.749	14.160	15.325	16.500	17.675	18.850	19.128	20.105	20.403	22.000	23.250	24.500	25.750	27.000	28.250
0.375	11.246	12.551	13.405	14.048	14.567	15.003	16.168	17.343	18.518	19.693	19.981	20.958	21.266	23.000	24.250	25.500	26.750	28.000	29.250
0.400	11.876	13.250	14.149	14.828	15.376	15.838	17.003	18.178	19.353	20.528	20.826	21.803	22.121	24.000	25.250	26.500	27.750	29.000	30.250
0.425	12.502	13.941	14.886	15.600	16.177	16.663	17.828	19.003	20.178	21.353	21.651	22.628	22.956	25.000	26.250	27.500	28.750	30.000	31.250
0.450	13.122	14.627	15.616	16.365	16.970	17.481	18.646	19.821	21.006	22.191	22.499	23.476	23.814	26.000	27.250	28.500	29.750	31.000	32.250
0.475	13.737	15.306	16.340	17.122	17.756	18.291	19.456	20.641	21.826	23.011	23.329	24.306	24.654	27.000	28.250	29.500	30.750	32.000	33.250
0.500	14.348	15.991	17.057	17.874	18.535	19.093	20.258	21.443	22.628	23.813	24.131	25.108	25.466	28.000	29.250	30.500	31.750	33.000	34.250
0.525	14.955	16.650	17.769	18.619	19.308	19.890	21.055	22.240	23.425	24.610	24.938	25.915	26.273	29.000	30.250	31.500	32.750	34.000	35.250
0.550	15.558	17.315	18.476	19.358	20.075	20.680	21.845	23.020	24.205	25.390	25.718	26.695	27.053	30.000	31.250	32.500	33.750	35.000	36.250
0.575	16.157	17.975	19.177	20.092	20.836	21.464	22.629	23.814	25.009	26.194	26.522	27.509	27.867	31.000	32.250	33.500	34.750	36.000	37.250
0.600	16.753	18.631	19.874	20.821	21.591	22.242	23.407	24.592	25.777	26.962	27.290	28.277	28.635	32.000	33.250	34.500	35.750	37.000	38.250
0.625	17.346	19.284	20.567	21.546	22.342	23.016	24.181	25.366	26.551	27.736	28.064	29.051	29.409	33.000	34.250	35.500	36.750	38.000	39.250
0.650	17.935	19.932	21.256	22.265	23.088	23.784	24.949	26.134	27.319	28.504	28.832	29.819	30.177	34.000	35.250	36.500	37.750	39.000	40.250
0.675	18.522	20.540	21.940	22.981	23.829	24.547	25.712	26.897	28.082	29.267	29.595	30.582	30.940	35.000	36.250	37.500	38.750	40.000	41.250
0.700	19.105	21.218	22.621	23.692	24.566	25.306	26.471	27.656	28.841	29.999	30.327	31.314	31.672	36.000	37.250	38.500	39.750	41.000	42.250
0.725	19.686	21.856	23.298	24.399	25.298	26.060	27.225	28.410	29.605	30.790	31.118	32.105	32.463	37.000	38.250	39.500	40.750	42.000	43.250
0.750	20.265	22.492	23.971	25.103	26.027	26.810	28.005	29.200	30.395	31.580	31.908	32.895	33.253	38.000	39.250	40.500	41.750	43.000	44.250
0.775	20.841	23.124	24.641	25.803	26.751	27.556	28.751	29.946	31.141	32.336	32.664	33.651	34.009	39.000	40.250	41.500	42.750	44.000	45.250
0.800	21.414	23.753	25.308	26.499	27.472	28.299	29.494	30.689	31.884	33.074	33.402	34.389	34.747	40.000	41.250	42.500	43.750	45.000	46.250
0.825	21.985	24.379	25.972	27.192	28.190	29.037	30.232	31.427	32.622	33.817	34.145	35.132	35.490	41.000	42.250	43.500	44.750	46.000	47.250
0.850	22.554	25.003	26.633	27.882	28.904	29.772	31.000	32.195	33.390	34.585	34.913	35.900	36.258	42.000	43.250	44.500	45.750	47.000	48.250
0.875	23.121	25.625	27.292	28.569	29.615	30.504	31.699	32.894	34.089	35.284	35.612	36.600	36.958	43.000	44.250	45.500	46.750	48.000	49.250
0.900	23.686	26.244	27.947	29.253	30.323	31.232	32.428	33.623	34.818	36.008	36.336	37.323	37.681	44.000	45.250	46.500	47.750	49.000	50.250
0.925	24.249	26.860	28.600	29.934	31.027	31.957	33.153	34.348	35.543	36.738	37.066	38.053	38.411	45.000	46.250	47.500	48.750	50.000	51.250
0.950	24.810	27.474	29.250	30.613	31.729	32.679	33.808	34.993	36.188	37.383	37.711	38.700	39.058	46.000	47.250	48.500	49.750	51.000	52.250
0.975	25.369	28.086	29.898	31.288	32.428	33.398	34.583	35.768	36.953	38.148	38.476	39.463	39.821	47.000	48.250	49.500	50.750	52.000	53.250
1.000	25.927	28.696	30.543	31.962	33.124	34.114	35.303	36.498	37.693	38.888	39.216	40.203	40.561	48.000	49.250	50.500	51.750	53.000	54.250
1.025	26.482	29.304	31.186	32.632	33.818	34.827	36.000	37.201	38.396	39.591	39.919	40.906	41.264	49.000	50.250	51.500	52.750	54.000	55.250
1.050	27.036	29.910	31.827	33.300	34.509	35.538	36.537	37.732	38.927	40.126	40.454	41.441	41.800	50.000	51.250	52.500	53.750	55.000	56.250
1.075	27.588	30.514	32.466	33.966	35.197	36.246	37.441	38.636	39.831	41.021	41.349	42.336	42.695	51.000	52.250	53.500	54.750	56.000	57.250
1.100	28.139	31.116	33.103	34.630	35.883	36.951	38.146	39.341	40.536	41.731	42.059	43.046	43.405	52.000	53.250	54.500	55.750	57.000	58.250
1.125	28.688	31.716	33.737	35.291	36.567	37.654	38.859	40.054	41.249	42.444	42.772	43.759	44.118	53.000	54.250	55.500	56.750	58.000	59.250
1.150	29.236	32.311	34.370	35.950	37.248	38.355	39.563	40.762	41.957	43.142	43.470	44.457	44.816	54.000	55.250	56.500	57.750	59.000	60.250
1.175	29.782	32.911	35.001	36.608	37.927	39.033	40.151	41.456	42.655	43.750	44.078	45.065	45.424	55.000	56.250	57.500	58.750	60.000	61.250
1.200	30.327	33.506	35.629	37.263	38.604	39.749	40.751	41.643	42.848	43.943	44.271	45.258	45.617	56.000	57.250	58.500	59.750	61.000	62.250
1.225	30.871	34.100	36.256	37.916	39.279	40.443	41.461	42.368	43.463	44.558	44.886	45.873	46.232	57.000	58.250	59.500	60.750	62.000	63.250
1.250	31.413	34.692	36.882	38.567	39.952	41.134	42.169	43.091	44.194	45.289	45.617	46.604	46.963	58.000	59.250	60.500	61.750	63.000	64.250
1.275	31.953	35.282	37.505	39.216	40.623	41.824	42.875	43.812	44.658	45.753	46.081	47.068	47.427	59.000	60.250	61.500	62.750	64.000	65.250
1.300	32.493	35.871	38.127	39.864	41.292	42.511	43.579	44.531	45.376	46.471	46.800	47.788	48.147	60.000	61.250	62.500	63.750	65.000	66.250
1.325	33.031	36.458	38.747	40.510	41.959	43.197	44.281	45.247	46.120	47.215	47.544	48.531	48.890	61.000	62.250	63.500	64.750	66.000	67.250
1.350	33.568	37.044	39.365	41.154	42.624	43.880	44.980	45.961	46.848	47.943	48.272	49.259	49.618	62.000	63.250	64.500	65.750	67.000	68.250
1.375	34.104	37.628	39.982	41.796	43.287	44.562	45.678	46.674	47.574	48.669	49.003	49.990	50.349	63.000	64.250	65.500	66.750	68.000	69.250
1.400	34.639	38.211	40.598	42.436	43.949	45.241	46.374	47.384	48.279	49.374	49.708	50.695	51.044	64.000	65.250	66.500	67.750	69.000	70.250
1.425	35.172	38.792	41.212	43.075	44.609	45.919	47.068	48.092	49.019	49.865	50.199	51.186	51.535	65.000	66.250	67.500	68.750	70.000	71.250