GENERALIZED LC MULTIPLE RESONANCE NETWORKS

Antonio Carlos M. de Queiroz

COPPE/EE - Electrical Engineering Program, Federal University of Rio de Janeiro
CP 68504, 21945-970 Rio de Janeiro, RJ, Brazil

ABSTRACT

This paper generalizes the idea of "multiple resonance networks" to energy transfers between inductors and between capacitors and inductors, extending the theory about this class of LC lossless circuits. The networks discussed have the property of transferring all the energy initially stored in a particular capacitor or inductor in an LC network to another element in the network, through a linear transient.

I. INTRODUCTION

The concept of "multiple resonance networks" [1] generalizes the concepts of the "double resonance" and "triple resonance" networks found in the literature [2][3]. In these references, the networks are composed of lossless inductors, transformers, and capacitors, and designed in such a way that all the energy initially stored in a capacitor is, at the closing of a switch, transferred to another capacitor in the circuit through a linear transient. At a certain instant, all the energy is concentrated in this other capacitor, and can be used for some purpose. Networks with this function can be found in pulsed power systems for physics research, where the energy is transferred from a large capacitance charged at low voltage to a small capacitance, that becomes charged at high voltage, with the same energy. The double resonance case is long known [2], having been used in early radio transmitters, electrotherapeutic devices, and in the generation of high voltages for demonstrations about electricity, as the "Tesla coil". The usual system is shown in fig. 1.

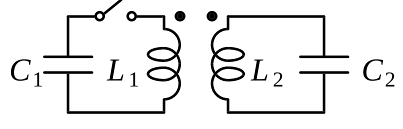

Fig. 1. Double resonance network with a transformer.

The secondary capacitor is usually formed by the distributed capacitance of the secondary windings of the transformer and the distributed capacitance of a terminal electrode. From the formulation in [1], the relations for optimum design can be obtained as:

$$L_1C_1 = L_2C_2$$
$$k_{12} = \frac{l^2 - k^2}{l^2 + k^2} \tag{1}$$

The constants k and l are two positive integers, with l-k odd, that define the operation "mode". They determine the two natural oscillation frequencies of the network with the switch closed, that appear as:

$$\omega_1 = k\omega_0; \quad \omega_2 = l\omega_0; \quad \omega_0 = \frac{1}{kl}\sqrt{\frac{k^2 + l^2}{2L_2C_2}} \tag{2}$$

The triple resonance system [3] was developed more recently, for pulsed power applications, as a solution for the insulation problem caused by the transformer in the double resonance system. As shown in fig. 2, a third inductor and another capacitor were added, with the result being that at the end of the energy transfer transient all the output voltage is over the third inductor, with no voltages or currents in the transformer.

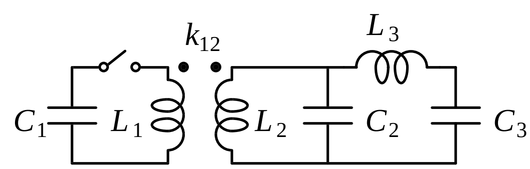

***Fig. 2. Triple resonance network with a transformer and additional inductor and capacitor*.**

The design formulas for the triple resonance network [1][4] can be arranged in convenient way as:

$$L_1C_1 = (L_2 + L_3)C_3$$
$$\frac{L_2}{L_3} = \frac{(l^2 - m^2)(k^2 - l^2)}{2k^2m^2}$$
$$k_{12} = \sqrt{\frac{L_2}{L_2 + L_3}} \tag{3}$$
$$\frac{C_2}{C_3} = \frac{2l^4}{(l^2 - m^2)(k^2 - l^2)}$$

where k, l, and m are three successive integers with odd differences, that define the operating mode, and also multiply a basic frequency ω_0 to produce the three natural oscillation frequencies of the complete circuit with the switch closed:

$$\omega_1 = k\omega_0; \quad \omega_2 = l\omega_0; \quad \omega_3 = m\omega_0$$
$$\omega_0 = \frac{1}{l}\sqrt{\frac{1}{L_3 C_3}} \qquad (4)$$

With the formulas shown, complete energy transfer occurs π/ω_0 seconds after the closure of the switch.

An example of the waveforms in the triple resonance case is shown in fig. 3. An initial voltage of 10 kV in C_1 is converted into 100 kV in C_3 after 50 µs. All the currents are null at this instant.

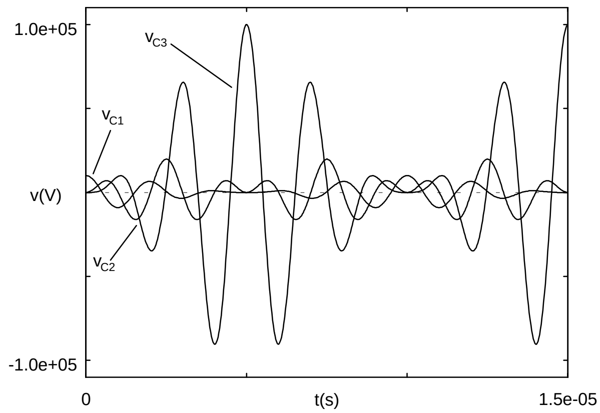

Fig. 3. Typical triple resonance network, and voltages in C_1, C_2, and C_3 after the closure of the switch. C_1=1 nF, C_2=126.3 pF, C_3=10 pF, L_1=110 µH, L_2=780.7 µH, L_3=10.13 mH, k_{12}=0.28131.

In [1][4], a generalized design procedure for this class of circuit was presented, that extends the possible designs for circuits of any even order. The procedure was based on the synthesis of the impedance seen from the output side of the network, in a version without a transformer (Fig. 4). A transformer can be always introduced later by a simple transformation.

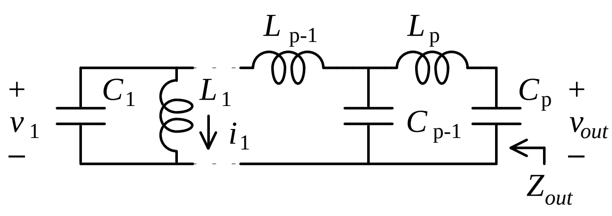

Fig. 4. Transformerless multiple resonance network.

All the cases considered involved energy transfer between two capacitors, placed at the extremities of the structure. In this work, it is shown how the same ideas in [1][4] can be extended to treat the cases of energy transfer between two inductors, and between a capacitor and an inductor.

II. ENERGY TRANSFER BETWEEN CAPACITORS

This is the case already studied in [1] and [4], resumed here for convenience. As the waveforms are symmetrical in respect to the time, the network can be designed as seen from the output side. An initial energy in C_p also makes its way back to C_1 in π/ω_0 seconds. The fundamental consideration is that a charged C_p can be replaced by an uncharged capacitor in parallel with an impulsive current source. In this configuration, the Laplace transform of the voltage over C_p becomes proportional to the impedance seen between the terminals of C_p. This impedance can be expressed in Foster's first form, as:

$$Z_{out}(s) \propto V_{out}(s) = \sum_{j=1}^{p} \frac{A_j s}{s^2 + k_j^2 \omega_0^2} \qquad (5)$$

Where the k_j correspond to the k, l, m, … of the simpler cases, sequences of positive integers with odd differences. A set of equations was then derived for the positive residues A_j, by the observation that the condition of all the capacitors having zero voltages, except C_1, at $t=\pi/\omega_0$, results in the output voltage and its first even derivatives, up to order $2p$-4, being null at this instant. The condition that resulted is:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & (-1)^{p-1} \\ k_1^2 & -k_2^2 & k_3^2 & \cdots & k_p^2(-1)^{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_1^{2p-4} & -k_2^{2p-4} & k_3^{2p-4} & \cdots & k_p^{2p-4}(-1)^{p-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_p \end{bmatrix} = \begin{bmatrix} 1/C_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (6)$$

The first equation comes from the behavior of (5) as $s\rightarrow\infty$, where it must reduce to a capacitance C_p. From the residues, Z_{out} can be obtained and expanded in Cauer's first form.

In [4], a different procedure was presented, starting from the observation that if an initial voltage $v_1(0)$ is applied to the input capacitor C_1, the output voltage has the expression:

$$V_{out}(s) = \frac{sC_1 L_1 v_1(0)\omega_0^{2p} k_1^2 k_2^2 \cdots k_p^2}{(s^2 + k_1^2\omega_0^2)(s^2 + k_2^2\omega_0^2)\cdots(s^2 + k_p^2\omega_0^2)} \qquad (7)$$

The denominator is known, and the numerator is a constant multiplying "s". If the numerator constant is chosen arbitrarily, and eq. (7) is expanded in partial fractions, the result is proportional to eq. (5), but with the origin of the time shifted by π/ω_0 seconds. The residues in eq. (5) can be obtained by simply taking the absolute values of the residues obtained from eq. (7), and scaling them so they add to $1/C_p$.

III. ENERGY TRANSFER BETWEEN INDUCTORS

This case is trivial. The network is designed exactly as in the case of energy transfer between capacitors, and converted into its dual form (fig. 5), adequately scaled in impedance. In the transformed network, and initial energy in one of the inductors at the ends of the structure is transferred to the inductor at the other side in π/ω_0 seconds. A transformer can be inserted at the right end. Note that the structure is the same of the first case, inverted, if L'_1 and C'_1 are interchanged.

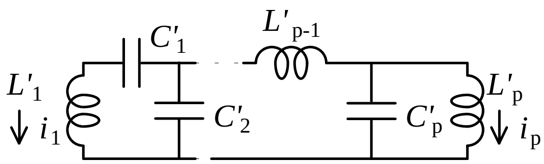

Fig. 5. multiple resonance network for energy transfer between inductors. $L'_1=\alpha C_1$, $C'_2=L_2/\alpha$, etc.

IV. ENERGY TRANSFER BETWEEN AN INDUCTOR AND A CAPACITOR

This case results in a structure that is exactly as in the other cases, and can be designed in a similar way, with some differences in the procedure. Considering again the transfer from the output capacitor to the first inductor (fig.4), eq. (5) still holds, but the energy transfer now occurs in $t=\pi/(2\omega_0)$ seconds, because the current in the first inductor becomes a sum of pure sinusoids instead of a sum of cosinusoids. Equations for the residues are obtained as follows: The sum of the residues continues to add to $1/C_p$, and so the first equation is the same. At $t=\pi/(2\omega_0)$, the currents in all the inductors, except L_1, are null. This condition results in the first odd derivatives of the output voltage being null at this time. A system of equations that corresponds to this is:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ k_1 & -k_2 & k_3 & \cdots & k_p(-1)^{p-1} \\ k_1^{\,3} & -k_2^{\,3} & k_3^{\,3} & \cdots & k_p^{\,3}(-1)^{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_1^{\,2p-3} & -k_2^{\,2p-3} & k_3^{\,2p-3} & \cdots & k_p^{\,2p-3}(-1)^{p-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_p \end{bmatrix} = \begin{bmatrix} 1/C_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (8)$$

The alternance of signs between the columns in the matrix is consequence of how the multipliers k_j are chosen. In this case, they must be all odd integers, with differences that are double odd numbers, so terms in the form $\sin k_j\pi/2$ have alternate signs for successive k_j. Note that this is precisely the complement of the rule used in the selection of the multipliers in the case of energy transfer between capacitors.

A simpler method can also be used in this case, starting from the observation that if a step current source I_{in} is connected in parallel with L_1 in fig. 4, the output voltage has a form similar to eq. (7), but the numerator is just a constant The constant can be arbitrarily chosen, and the expression expanded in partial fractions, but in this case considering sinusoids at the output instead of cosinusoids in the expansion, or terms in the form $B_j k_j\omega_0/(s^2+(k_j\omega_0)^2)$. The correct residues A_j for the generation of the output impedance are obtained from the absolute values of the B_j, scaled so they add to $1/C_p$, as in the first case.

V. EXAMPLES

Example 1

Consider first a double resonance circuit, with energy transfer from L_1 to C_2. Let the multipliers be $k=k_1$ and $l=k_2$, $\omega_0=1$, and $C_p=C_2=1$. The system of equations (8) results in:

$$A_1 = \frac{l}{l+k}; \quad A_2 = \frac{k}{l+k} \quad (9)$$

The output impedance, and the element values (structure in fig. 4) are obtained as:

$$Z_{out}(s) = \frac{s^3+(k^2-kl+l^2)s}{s^4+(k^2+l^2)s^2+k^2l^2}$$
$$C_2 = 1; \quad L_2 = \frac{1}{kl}; \quad C_1 = \frac{kl}{(l-k)^2}; \quad L_1 = \frac{(l-k)^2}{k^2l^2} \quad (10)$$

Note that the structure is antimmetrical, and can be scaled in impedance so:

$$C_2 = L_1 = \frac{l-k}{kl}; \quad C_1 = L_2 = \frac{1}{l-k} \quad (11)$$

A transformer can be included by first inserting an ideal transformer between C_1 and L_1 and then converting its combination with L_1 and L_2 into a real transformer [1][4]. The result is a structure as fig. 1 (without the switch), where:

$$L_1C_1 = \frac{1}{kl}; \quad L_2C_2 = \frac{k^2-kl+l^2}{k^2l^2};$$
$$k_{12} = \sqrt{\frac{(l-k)^2}{(l-k)^2+kl}} \quad (12)$$

It is interesting to observe that the two coupled LC tanks are not to be tuned to the same frequency, as happens in the classic case (eqs (1)).

As a numerical example, consider the circuit obtained from the insertion of a transformer with turns ratio 10 in the transformerless circuit obtained from eqs. (10), in mode k=3, l=5, followed by an impedance scaling to make C_2=100 pF and a frequency scaling to make ω_0=100000 rads/s. The result is C_1=37.5 nF, C_2=100 pF, L_1=0.1777 mH, L_2=84.444 mH, and k_{12}=0.4588. Fig. 6 shows the voltage and current waveforms obtained, starting with a current of 1 A in L_1. At $t=\pi/2\text{x}10^{-5}$ seconds, the energy transfer to C_2 is complete.

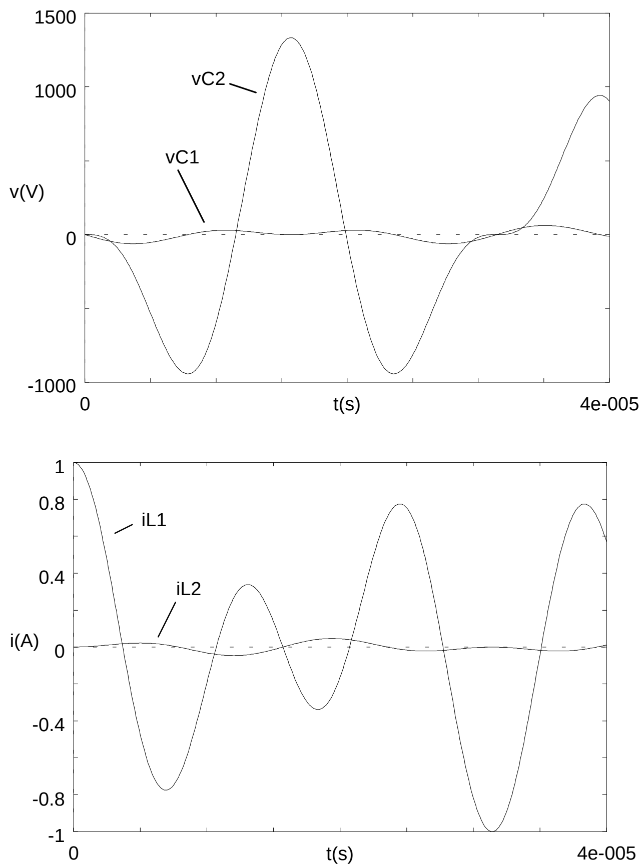

Fig. 6. Energy transfer between L_1 and C_2 in a double resonance network with a transformer.

Example 2

For the triple resonance case with energy transfer between L_1 and C_3, the transformerless network obtained from eqs. (8) and the expansion of eq. (5) is also antimmetrical (this happens in all cases, because the dual circuit also transfers energy between an inductor and a capacitor). With ω_0 normalized to 1 and a convenient impedance scaling to force the antimmetry, the element values are obtained (for the circuit in fig. 4 again) as eqs. (13). The multipliers $k=k_1$, $l=k_2$, $m=k_3$ must be all odd, with double odd differences (ex: (1,3,5), (3,5,7), (3,5,11), etc.) for correct results. Energy transfer occurs in $\pi/2$ seconds, with 1 A at L_1 becoming 1 V at C_3.

A transformer can be inserted at the left side, resulting a circuit as in fig. 2, without the switch. With this a set of design equations are obtained as eqs. (14). Note that the two LC circuits at the ends of the structure become tuned to the same frequency, what doesn't happen in the case of energy transfer between capacitors (eqs. (3)).

$$
\begin{aligned}
L_1 = C_3 &= \frac{(l-m)(k^2+k(m-l)-lm)}{klm(k-l+m)};\\
C_1 = L_3 &= \frac{(k-l+m)^2}{(k^2+k(m-l)-lm)(l-m)};\\
L_2 = C_2 &= \frac{1}{k-l+m}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&C_1L_1 = C_3L_3 = \frac{k-l+m}{klm};\\
&C_2L_2 = \frac{k(l-m)+lm}{klm(k-l+m)};\\
&\frac{C_2}{C_3} = \frac{klm}{(l-m)(k^2+k(m-l)-lm)}\\
&k_{12} = \sqrt{\frac{(l-m)(k^2+k(m-l)-lm)}{k^2(l-m)-k(l^2-3lm+m^2)-lm(l-m)}}
\end{aligned}
\tag{14}
$$

VI. CONCLUSIONS

The concept of multiple resonance networks was extended to the cases of energy transfer involving inductors. A classical induction coil, still used in ignition systems for motors, is an example where the double resonance case can find application. A triple resonance circuit would have the interesting property of removing all the voltages and currents from the transformer at the end of the energy transfer process. The techniques presented can be directly extended for higher-order circuits, although it is not clear if they would be more than curiosities.

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