The triple resonance network with sinusoidal excitation

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***Abstract*—In a "triple resonance network" operating with zero-state response, the network is composed of a transformer powered through a primary capacitor from a voltage source that produces a burst of sine cycles, that has at its output side a shunt capacitor and a third coil, whose other end connects to a capacitive load. The objective is, at the end of a number of cycles, to have all the energy in the system concentrated at the load capacitance. The operation with the zero-state response allows the charging of the output from a much smaller input voltage than what would be required in a capacitor-discharge system, operating with the zero-input response. In the comparison with a similar "double resonance" network, a fundamental limitation in the voltage gain is identified, and alternative solutions are proposed. These networks find applications in high-voltage generators for pulsed power applications, and their design procedure uses curious aspects of the circuit theory of lossless linear circuits.**

I. INTRODUCTION

Triple resonance networks (Fig. 1b) were introduced in [2][3][4], as a variation of the double resonance network, or Tesla transformer (Fig. 1a), with the purpose of charging rapidly a relatively small capacitive load to high voltage. The double resonance circuit is composed of a primary capacitor C_1, charged to moderate high voltage, that is connected to the input inductor L_a of a transformer with low coupling coefficient k_{ab} by a switch. The system enters then a transient involving two sinusoidal oscillations, that with proper design ideally transfers all the input capacitor energy to the output load capacitor $C_L = C_2$. Energy conservation dictates that the maximum voltage gain is:

$$\frac{v_{out(peak)}}{v_{C_1(peak)}} = \sqrt{\frac{C_1}{C_L}} \quad (1)$$

The triple resonance version introduces a third inductor L_3 connecting C_2 to the load $C_L = C_3$. With this the high-voltage end can be moved away from the transformer, and it is possible to obtain faster energy transfer, by using a transformer with increased coupling coefficient, now possible due to the reduced insulation requirements. The transient after the closing of the switch involves oscillations at three frequencies. The structure is sometimes cited as Tesla "magnifier". The design procedure described in [1], and in an improved version in [5], departs from the ratio of the three resonance frequencies of the network, k:l:m, and from some given component values. In any case, energy conservation fixes the maximum voltage gain as (1).

In [6], a different mode of operation was described, where instead of a charged primary capacitor, the source of energy for the system is a sinusoidal voltage source connected in series with C_1 (Fig 2) (a square wave can also be used, with just a small error). The output voltage has three (for Fig. 2a) or four (for Fig. 2b) frequency components, one coming from the input and the others from the natural frequencies of the network. The examples in [6] were all for the double resonance case. In this paper, the triple resonance network with sinusoidal excitation is discussed.

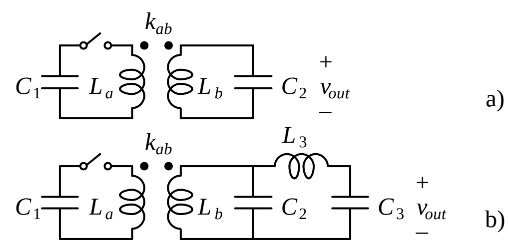

Figure 1. Double resonance (a) and triple resonance (b) networks.

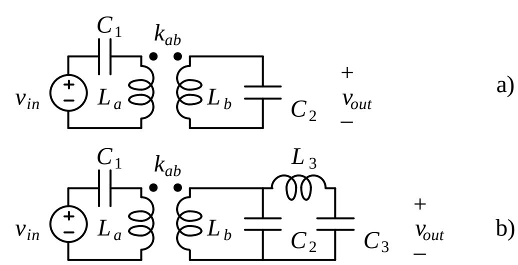

Figure 2. The same networks operating with zero-state response.

II. DESIGN OF THE TRIPLE RESONANCE NETWORK EXCITED BY A SINUSOIDAL VOLTAGE SOURCE

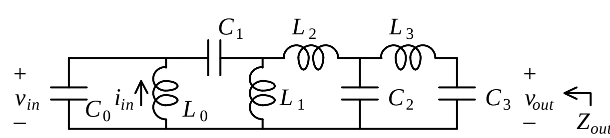

Figure 3. Transformerless triple resonance network excited by a low-impedance LC tank L_0C_0.

The derivation follows the same ideas discussed in [6], departing from the transformerless 8th-order network shown in Fig. 3, excited by an initial current i_{in} in the first inductor L_0. Considering that the structure places three transmission zeros at 0 between an input current $I_{in}(s) = i_{in}/s$ in parallel with L_0 and the output v_{out}, $V_{out}(s)$ has the form:

$$V_{out}(s)=\frac{\alpha s^2}{(s^2+k^2)(s^2+l^2)(s^2+m^2)(s^2+n^2)}= \tag{2}$$
$$=\frac{B_1k}{(s^2+k^2)}+\frac{B_2l}{(s^2+l^2)}+\frac{B_3m}{(s^2+m^2)}+\frac{B_4n}{(s^2+n^2)}$$

where k, l, m, and n are factors that multiply a common frequency ω_0, here normalized to $\omega_0 = 1$. The ratio k:l:m:n then defines the mode of operation of the network. If these factors are successive integers with differences that are doubles of odd integers, as 1:3:5:7, 3:5:7:9, 3:5:7:13, etc., all the four sinusoids that form the output voltage add constructively at t = $\pi/2$, when the output voltage is maximum. Changing the origin of time to this instant, v_{out} becomes a sum of cosinusoids. Considering that all the energy in the circuit is concentrated at C_3, v_{out} can be generated by an impulsive current source in parallel with C_3, and $V_{out}(s)$ is proportional to the output impedance of the network. With a proper normalization, $V_{out}(s) = Z_{out}(s)$:

$$V_{out}(s)=Z_{out}(s)=\frac{|B_1|s}{(s^2+k^2)}+\frac{|B_2|s}{(s^2+l^2)}+\frac{|B_3|s}{(s^2+m^2)}+\frac{|B_4|s}{(s^2+n^2)} \tag{3}$$

The residues B_i can be found, except for an arbitrary multiplying factor, by expanding (2) with α = 1. For convenience, the B_i can be scaled so their absolute values add to 1. The results are:

$$B_1=-\frac{k(m-n)(l+n)(l-m)}{(k-l+m-n)(k+l)(k-m)(k+n)};B_2=\frac{l(m-n)(k-n)(k+m)}{(k-l+m-n)(k+l)(l+m)(l-n)} \tag{4}$$
$$B_3=-\frac{m(l+n)(k-n)(k-l)}{(k-l+m-n)(k-n)(l+m)(m+n)};B_4=\frac{n(l-m)(k+m)(k-l)}{(k-l+m-n)(k+n)(l-n)(m+n)}$$

The expansion of (3) with the residues (4) in the structure in Fig. 3 would result in a "quadruple resonance network", with complete energy transfer from L_0 to C_3 in $\pi/2$ seconds. Considering now that when the output voltage is maximum there is a voltage v_{in} left at C_0, a voltage generated by an impulsive current source in parallel with C_0 is added to v_{out}. The contribution of v_{in} in v_{out} has the form:

$$V_{out}'(s)=\frac{\alpha' s^3}{(s^2+k^2)(s^2+l^2)(s^2+m^2)(s^2+n^2)}= \tag{5}$$
$$=\frac{A_1s}{(s^2+k^2)}+\frac{A_2s}{(s^2+l^2)}+\frac{A_3s}{(s^2+m^2)}+\frac{A_4s}{(s^2+n^2)}$$

The residues A_i can be obtained by expanding (5) with α' = 1, and, for convenience, scaling the residues so their absolute values add to 1. The results are:

$$A_1=-\frac{k^2(m^2-n^2)(l^2-m^2)}{2(k^2-m^2)(k^2m^2-l^2n^2)};A_2=\frac{l^2(m^2-n^2)(k^2-n^2)}{2(l^2-n^2)(k^2m^2-l^2n^2)} \tag{6}$$
$$A_3=-\frac{m^2(k^2-n^2)(k^2-l^2)}{2(k^2-m^2)(k^2m^2-l^2n^2)};A_4=\frac{n^2(l^2-m^2)(k^2-l^2)}{2(l^2-n^2)(k^2m^2-l^2n^2)}$$

Introducing a factor ε to control how important is this contribution, $V_{out}(s)$ is given by:

$$V_{out}(s)=\frac{|B_1|s}{(s^2+k^2)}+\frac{|B_2|s}{(s^2+l^2)}+\frac{|B_3|s}{(s^2+m^2)}+\frac{|B_4|s}{(s^2+n^2)}= \tag{7}$$
$$=Z_{out}(s)+\varepsilon\left(\frac{A_1s}{(s^2+k^2)}+\frac{A_2s}{(s^2+l^2)}+\frac{A_3s}{(s^2+m^2)}+\frac{A_4s}{(s^2+n^2)}\right)$$

And so, the output impedance of the network can be obtained as:

$$Z_{out}(s)=\frac{(|B_1|-\varepsilon A_1)s}{s^2+k^2}+\frac{(|B_2|-\varepsilon A_2)s}{s^2+l^2}+\frac{(|B_3|-\varepsilon A_3)s}{s^2+m^2}+\frac{(|B_4|-\varepsilon A_4)s}{s^2+n^2} \tag{8}$$

There are four values of ε that cause one of the terms to disappear. When this happens, the tank L_0C_0 becomes a zero-impedance LC tank, that acts as an ideal sinusoidal voltage source driving the network. These values are:

$$\varepsilon_1=-\frac{B_1}{A_1};\varepsilon_2=\frac{B_2}{A_2};\varepsilon_3=-\frac{B_3}{A_3};\varepsilon_4=\frac{B_4}{A_4} \tag{9}$$

Using ε_1, the driving signal is at the first frequency, normalized to ω = k. The output impedance has the expansion:

$$Z_{out1}=\frac{(B_2-\varepsilon_1A_2)s}{s^2+l^2}+\frac{(-B_3-\varepsilon_1A_3)s}{s^2+m^2}+\frac{(B_4-\varepsilon_1A_4)s}{s^2+n^2} \tag{10}$$
$$B_2-\varepsilon_1A_2=\frac{l(m-n)(k-n)(k+m)}{k(k-l+m-n)(l+m)(l-n)}$$
$$-B_3-\varepsilon_1A_3=\frac{m(l+n)(k-n)(k-l)}{k(k-l+m-n)(l+m)(m+n)}$$
$$B_4-\varepsilon_1A_4=\frac{n(l-m)(k+m)(k-l)}{k(k-l+m-n)(l-n)(m+n)}$$

Using ε_2, the excitation is at $\omega = l$:

$$Z_{out2}=\frac{(-B_1-\varepsilon_2A_1)s}{s^2+k^2}+\frac{(-B_3-\varepsilon_2A_3)s}{s^2+m^2}+\frac{(B_4-\varepsilon_2A_4)s}{s^2+n^2} \tag{11}$$
$$-B_1-\varepsilon_2A_1=\frac{k(m-n)(l+n)(l-m)}{l(k-l+m-n)(k-m)(k+n)}$$
$$-B_3-\varepsilon_2A_3=\frac{m(l+n)(k-n)(k-l)}{l(k-l+m-n)(k-m)(m+n)}$$
$$B_4-\varepsilon_2A_4=\frac{n(l-m)(k+m)(k-l)}{l(k-l+m-n)(k+n)(m+n)}$$

Using ε_3, the excitation is at $\omega = m$:

$$Z_{out3}=\frac{(-B_1-\varepsilon_3A_1)s}{s^2+k^2}+\frac{(B_2-\varepsilon_3A_2)s}{s^2+l^2}+\frac{(-B_4-\varepsilon_3A_4)s}{s^2+n^2} \tag{12}$$
$$-B_1-\varepsilon_3A_1=-\frac{k(m-n)(l+n)(l-m)}{m(k-l+m-n)(k+l)(k+n)}$$
$$B_2-\varepsilon_3A_2=\frac{l(m-n)(k-n)(k+m)}{m(k-l+m-n)(k+l)(l-n)}$$
$$B_4-\varepsilon_3A_4=\frac{n(l-m)(k+m)(k-l)}{m(k-l+m-n)(k+n)(l-n)}$$

And using ε_4, the excitation is at $\omega = n$:

$$Z_{out4}=\frac{(-B_1-\varepsilon_4A_1)s}{s^2+k^2}+\frac{(B_2-\varepsilon_4A_2)s}{s^2+l^2}+\frac{(-B_3-\varepsilon_4A_3)s}{s^2+m^2} \tag{13}$$
$$-B_1-\varepsilon_4A_1=\frac{k(m-n)(l+n)(l-m)}{n(k-l+m-n)(k+l)(k-m)}$$
$$B_2-\varepsilon_4A_2=-\frac{l(m-n)(k-n)(k+m)}{n(k-l+m-n)(k+l)(l+m)}$$
$$-B_3-\varepsilon_4A_3=\frac{m(l+n)(k-n)(k-l)}{n(k-l+m-n)(k-m)(l+m)}$$

It's seen that only $Z_{out3}(s)$ and $Z_{out4}(s)$ are realizable, with all the residues always positive if $k<l<m<n$. This is a general property of these networks. Solutions exist only when the excitation corresponds to one of the two highest frequencies of the output signal. The final network is obtained by the expansion of $Z_{out}(s)$ with the structure of Fig 3, using Cauer's first form, followed by the conversion of L_1-L_2 into a transformer, and adequate impedance and frequency scaling [1][6]. Explicit formulas for the elements become too complex to be listed here, but it's easy to carry out the calculations numerically.

A. *Example 1:*

Consider the design of a network operating in mode 5:7:9:11, with driving signal at the normalized frequency $\omega = 9$ rad/s. Using the formulas (12), the result for $Z_{out}(s)$ and the normalized element values are:

$$Z_{out3}(s) = \frac{\frac{5}{96}s}{s^2+25} + \frac{\frac{49}{72}s}{s^2+49} + \frac{\frac{77}{288}s}{s^2+121} \tag{14}$$

$$C_3 = 1.000000000\text{ F}; L_3 = 0.014925373\text{ H}; C_2 = 4.110805861\text{ F}$$
$$L_2 = 0.002963489\text{ H}; C_1 = 126.654620404\text{ F}; L_1 = 0.000292956\text{ H}$$

Introducing a transformer and scaling the network so $C_1 = 5$ nF, $C_3 = 10$ pF, and $L_3 = 30$ mH, the final element values and the excitation frequency become:

$$C_3 = 10.00\text{ pF}; L_3 = 30.00\text{ mH}; C_2 = 41.11\text{ pF}; L_b = 6545\,\mu\text{H}$$
$$C_1 = 5.000\text{ nF}; L_a = 149.2\,\mu\text{H}; k_{ab} = 0.2999;\ f_{in} = 319.5\text{ kHz}$$

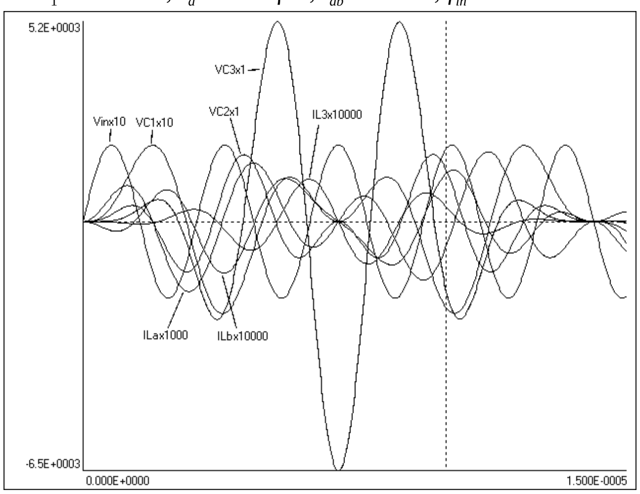

Figure 4. Voltages and currents with complete energy transfer

Fig. 4 shows the results of a simulation considering a sinusoid with 200 V of amplitude applied to the circuit. A problem with this design can be observed. The voltage gain is quite low, with the output reaching only 6.504 kV. Consider the same C_1 and C_3 used in a double resonance network (Fig. 2a, design formulas and voltage gains in [6]), operating in mode 5:7:9, excited at the central frequency. The output voltage reaches 15.00 kV with the same 200 V of input peak voltage. The triple resonance design above is more similar to the double resonance design with excitation at the upper frequency (m), mode 5:7:9, that results in only 4.998 kV of maximum output voltage. The triple resonance design with excitation at the upper frequency (n) is even worse, resulting in just 4.081 kV. The capacitor-discharge version would produce, from (1), 4.472 kV.

III. Incomplete Energy Transfer to Increase the Voltage Gain

A possibility that turns realizable the triple resonance network with excitation at the second frequency is to design for incomplete energy transfer, leaving some energy in the capacitors C_1 or C_2 when the output voltage is maximum.

A. *Energy left at C_1:*

Considerations similar to the ones that lead to (8) and (11), result in:

$$Z_{out2}(s) = \frac{(|B_1| - \varepsilon_2 A_1 - \gamma D_1)s}{s^2+k^2} + \frac{(|B_3| - \varepsilon_2 A_3 - \gamma D_3)s}{s^2+m^2} + \frac{(|B_4| - \varepsilon_2 A_4 - \gamma D_4)s}{s^2+n^2} \tag{15}$$
$$D_1 = \frac{m^2-n^2}{2(k^2-n^2)};\ D_3 = -\frac{1}{2};\ D_4 = \frac{k^2-m^2}{2(k^2-n^2)}$$

Where the D_i are the residues of the expansion in partial fractions of the contribution of an impulsive current source in parallel with C_1 to the output voltage, already considering the tank L_0C_0 reduced to a voltage source. The expansion is identical to the one used in the design of a capacitor-discharge triple resonance network [1]. The parameter γ can be always chosen so all the three residues of the expansion of $Z_{out2}(s)$ are positive, if it is between the limits:

$$\frac{|B_4| - \varepsilon_2 A_4}{D_4} > \gamma > \frac{|B_3| - \varepsilon_2 A_3}{D_3} \tag{16}$$

B. *Example 2:*

Consider again mode 5:7:9:11, but with excitation at $\omega = 7$ rad/s. Using (15), $Z_{out2}(s)$ results as:

$$Z_{out2}(s) = \frac{\left(\frac{175781}{875000} - \gamma\frac{5}{24}\right)s}{s^2+25} + \frac{\left(\frac{243}{280} + \gamma\frac{1}{2}\right)s}{s^2+81} + \frac{\left(-\frac{11}{160} - \gamma\frac{7}{24}\right)s}{s^2+121} \tag{17}$$

The limits (16) are then $\gamma > -243/140 = -1.736$ and $\gamma < -33/140 = -0.2357$. At the limits, one term of the expansion disappears, and the tank L_1C_1 too. A reasonable value is between the limits, $\gamma = -0.9$, what results, approximately, in the maximum output voltage, of 10.82 kV. The resulting element values are:

$$C_3 = 10.00\text{ pF}; L_3 = 30.00\text{ mH}; C_2 = 33.70\text{ pF}; L_b = 14.81\text{ mH}$$
$$C_1 = 5.000\text{ nF}; L_a = 54.17\,\mu\text{H}; k_{ab} = 0.4279;\ f_{in} = 248.5\text{ kHz}$$

Fig. 5 shows the resulting waveforms. The voltage gain is substantially increased, but significant energy (48.21 %) is left at C_1.

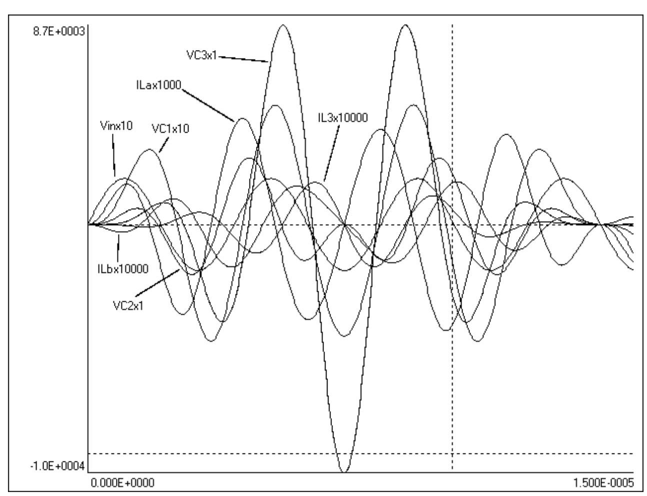

Figure 5. Voltages and currents when some energy is left at C_1.

C. Energy left at C_2:

The effect of a voltage v_2 at C_2 in v_{out} can be calculated by placing an impulsive current source $q = C_2 v_2$ in parallel with C_2, and shifting it to the output, as shown in fig. 6.

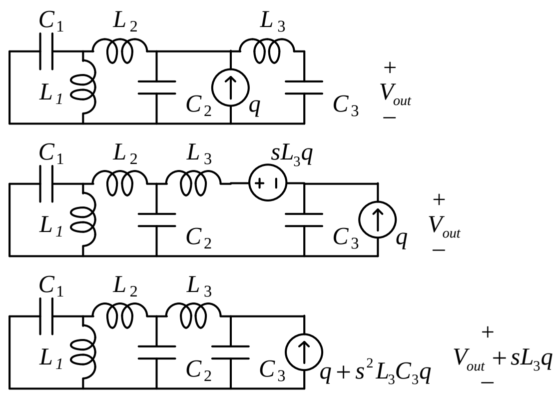

Figure 6. Finding the effect of a charge in C_2 at the output.

The output voltage can then be expressed in function of the output impedance, and expanded in partial fractions. After some manipulation, and using the fact that $E_1+E_3+E_4 = 1/C_3$, the result is:

$$\begin{aligned} V_{out} &= Z_{out}\left(q+s^2L_3C_3q\right)-sL_3q = \\ &= \left(\frac{E_1 s}{s^2+k^2}+\frac{E_3 s}{s^2+m^2}+\frac{E_4 s}{s^2+n^2}\right)\left(q+s^2L_3C_3q\right)-sL_3q = \\ &= \frac{E_1q\left(1-k^2\right)s}{s^2+k^2}+\frac{E_3q\left(1-m^2\right)s}{s^2+m^2}+\frac{E_4q\left(1-n^2\right)s}{s^2+n^2} \end{aligned} \tag{18}$$

Applying (18) in (7), using $\varepsilon = \varepsilon_2$, the output impedance can be found as:

$$\begin{aligned} Z_{out} &= \frac{E_1 s}{s^2+k^2}+\frac{E_3 s}{s^2+m^2}+\frac{E_4 s}{s^2+n^2} = \frac{\left(|B_1|-\varepsilon_2A_1-E_1q\left(1-k^2\right)\right)s}{s^2+k^2}+ \\ &+\frac{\left(|B_3|-\varepsilon_2A_3-E_3q\left(1-m^2\right)\right)s}{s^2+m^2}+\frac{\left(|B_4|-\varepsilon_2A_4-E_4q\left(1-n^2\right)\right)s}{s^2+n^2} \end{aligned} \tag{19}$$

And the residues E_i of the expansion of the output impedance are then:

$$E_1=\frac{|B_1|-\varepsilon_2A_1}{1+q\left(1-k^2\right)};\ E_3=\frac{|B_3|-\varepsilon_2A_3}{1+q\left(1-m^2\right)};\ E_4=\frac{|B_4|-\varepsilon_2A_4}{1+q\left(1-n^2\right)} \tag{20}$$

To make E_4 positive and leave E_3 positive too, q must be between the limits:

$$\frac{1}{n^2-1}<q<\frac{1}{m^2-1} \tag{21}$$

D. Example 3:

With the same parameters of example 2, the limits for q are $0.0125 > q > 0.00833$. The maximum voltage gain is obtained with $q = 0.0088$, approximately, that results in maximum output voltage of 9.68 kV:

$$\begin{aligned} &C_3 = 10.00\text{ pF}; L_3 = 30.00\text{ mH}; C_2 = 140.2\text{ pF}; L_b = 4151\ \mu\text{H} \\ &C_1 = 5.000\text{ nF}; L_a = 138.6\ \mu\text{H}; k_{ab} = 0.6001;\ f_{in} = 215.7\text{ kHz} \end{aligned}$$

The voltages and currents are shown in Fig. 7. The voltage gain is smaller than in the previous example, although the efficiency is slightly better. 45.96 % of the energy remains in C_2. It is also possible to combine the two transformations, leaving energy in C_1 and C_2, by applying first the transformation on $Z_{out}(s)$ that leaves energy in C_1. There is no apparent advantage on this, however.

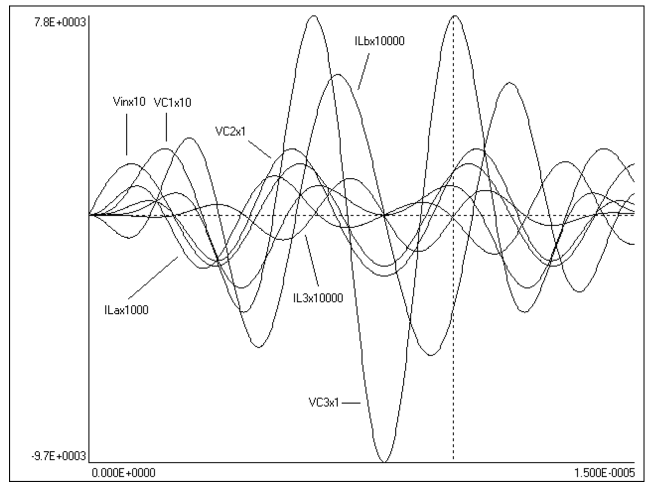

Figure 7. Voltages and currents when energy is left at C_2.

IV. CONCLUSIONS

The detailed design of the triple resonance network with sinusoidal excitation, assuming no losses, was described. It was verified that the obtainable voltage gain for fixed input and output capacitances and number of input cycles is significantly smaller than in a similar double resonance network. There is no solution for an arbitrarily small C_2, what would reduce the network to the double resonance case with excitation between the resonances, and so produce a similar high voltage gain. Two possible designs with incomplete energy transfer were then described, that result in larger voltage gain. A computer program that does all the calculations and plots is available [7].

REFERENCES

[1] A. C. M. de Queiroz, "Multiple resonance networks", IEEE Transactions on Circuits and Systems I, Vol. 49, No. 2, pp. 240-244, February 2002.

[2] F. M. Bieniosek, "Triple resonance high voltage pulse transformer circuit," Proc. 6th IEEE Pulsed Power Conference, pp. 700-702, 1987.

[3] F. M. Bieniosek, "Triple resonance pulse transformer circuit," Review of Scientific Instruments 61 (6) pp. 1717-1719, June 1990.

[4] F. M. Bieniosek, "Triple resonance pulse transformer circuit," U. S. patent No. 4833421, 23 May, 1989.

[5] A. C. M. de Queiroz, "A generalized approach to the design of multiple resonance networks", IEEE Transactions on Circuits and Systems I, Vol. 53, No. 4, April 2006, pp. 918-927.

[6] A. C. M. de Queiroz, "Multiple resonance networks with incomplete energy transfer and operating with zero-state response", 2005 IEEE International Symposium on Circuits and Systems, Kobe, Japan, May de 2005, Vol. I, pp. 236-239.

[7] A. C. M. de Queiroz, Program "trsstcd", available for download at http://www.coe.ufrj.br/~acmq/programs.